MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 108 No. 4

MEETING OF 1948 MAY 14

Professor W. M. H. Greaves, President, in the Chair

The election by the Council of the following Fellows was duly confirmed:—

Christine M. Bondi, B.A., 38 Trinity Street, Cambridge (proposed by F. Hoyle);

Norman Stewart Parks Bradbury, 94 Hadley Road, New Barnet, Herts (proposed by J. H. Finch);

Geoffrey Ronald Burbidge, B.Sc., 42 Leith Mansions, London, W. 9 (proposed by E. M. Peachey);

Eric Hugh Cooper, 74 Dewsbury Road, Dollis Hill, London, N.W. 10 (proposed by E. R. Osler);

Walter Cecil Fenwick, M.B.E., 67 Princes Road, Buckhurst Hill, Essex (proposed by H. R. Hulme); and

J. Charles Whitney Hardwick, 4 Hyde Road, Watford, Hertfordshire (proposed by E. W. Barlow).

One hundred and two presents were announced as having been received since the last meeting, including:—

D. F. Becker, *Einführung in die Astronomie*, presented by the author; and

Sen Gupta, *Ancient Indian Chronology illustrating some of the most important Astronomical Methods*, presented by Dr A. Hunter.

At a Special General Meeting of Fellows, held immediately preceding the Ordinary Meeting, the following resolution was proposed by the Treasurer on behalf of the Council:

That the Coleman 8-inch refractor belonging to the Society be sold to the Port Elizabeth Centre of the Astronomical Society of South Africa for the sum of seven hundred pounds.

The Treasurer explained the reasons which had led the Council to frame this resolution, which was then put to the meeting from the Chair and carried unanimously.

CONVERSAZIONE ON 1948 JULY 23

A Conversazione was held on the evening of July 23 when the opportunity was taken for the Society to entertain astronomers visiting England prior to the Assembly of the International Astronomical Union at Zürich, as well as representatives of other Scientific Societies and Institutions. During the evening cinematograph films of Solar Activity were shown in the Meeting Room, with a commentary given by Professor F. J. M. Stratton. Exhibits and demonstrations were arranged in the libraries ; documents and portraits connected with the centenary of the deaths of Caroline Herschel and of Father Francisco de Vico were on view.

STAR-STREAMING AND THE STABILITY OF SPIRAL ORBITS IN SPIRAL NEBULAE

I.—MOTION ROUND A POINT-NUCLEUS

E. A. Milne, F.R.S.

(Communicated by the Director of the University Observatory, Oxford)

(Received 1948 June 5)

Summary

It is shown that the spiral orbits determined in recent papers by the author for motion round a galactic nucleus are *stable*. It is further shown that small deviations from these orbits give rise to the phenomenon of star-streaming, just as on Lindblad's theory. The velocity ellipse relative to a non-rotating frame is calculated in terms of the velocity ellipse relative to a rotating frame, following Lindblad and Smart, and it is shown that the two axes of the ellipse interchange themselves when we pass from the latter ellipse to the former, just as on the classical theory of gravitation.

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1. *Introduction.*—In two recent papers * I have applied the kinematic theory of gravitation to obtain the equation of an arm of a spiral nebula. Using the t-scale of time, i. e. the scale in which the galaxies are receding from one another, I showed that the orbits of the particles or stars under the influence of a massive point-nucleus were the equiangular spirals

$$\theta-\theta_0=\left(\frac{c^3t_0^3}{r_0^3}\frac{M}{M_0}\right)^{\frac{1}{2}}\log\frac{t}{t_0}, \qquad r=r_0\frac{t}{t_0}, \tag{1), (1'}$$

where (r, θ) are current polar coordinates at time t reckoned from the natural zero of time, (r_0, θ_0) the present values of these coordinates at epoch $t=t_0$, t_0 being the present age of the universe, M the mass of the nebular nucleus and M_0 the mass of the equivalent homogeneous universe. The value of the constant of gravitation γ_t at time t is given by

$$\gamma_t=c^3t/M_0. \tag{2}$$

The *present positions* (r_0, θ_0) of the various particles or stars, as strung out in their respective spiral orbits, were shown to be given by the equation

$$\theta_0=\beta+\left(\frac{c^3t_0^3}{r_0^3}\frac{M}{M_0}\right)^{\frac{1}{2}}\log\frac{r_0}{r_1}, \tag{3}$$

where (r_1, β) is the fixed point through which the various particles were emitted at various times t_1. Further, it was shown that if the present positions (r_0, θ_0) given by (3) move according to the orbits (1), (1'), then the various positions at time t are given by

$$\theta=\beta+\left(\frac{c^3t^3}{r^3}\frac{M}{M_0}\right)^{\frac{1}{2}}\log\frac{r}{r_1}, \tag{4}$$

so that this is the permanent equation of a spiral arm at time t.

The equations (1), (1') correspond in τ-measure to circular motion, i. e. Keplerian motion under the central point-mass M. Two questions arise. First,

* *M.N.*, **106**, 180, 1946; *Ap. J.*, **106**, 137, 1947.

are these spiral orbits stable? Second, what is the generalization of the orbits and spiral arms to the case where the mass of the nebula is not considered as wholly concentrated in a point-nucleus? The present paper is concerned with the answer to the first of these questions. The succeeding paper will be concerned with the second question.

The question of the stability of the spiral orbits will be investigated by Lindblad's * method of using a rotating frame to which to refer the varied motion. I shall follow the version of Lindblad's method reproduced by Smart †, but adapting it to the kinematic theory of gravitation, so that I shall consider disturbances from spiral orbits instead of from circular orbits; it will be necessary to take into account the non-uniform rate of rotation of the rotating frame. It will be shown that the spiral orbits are stable, and that small variations from ideal spirals should give rise to the phenomenon of star-streaming, exactly as in Lindblad's theory. But the accompanying theory of the observed spiral appearance of galaxies is quite different from Lindblad's theory, first because the latter reposes on the classical Newtonian theory of gravitation, secondly because in Lindblad's account of spiral form, the arms are taken to be derived from circular orbits which have become *unstable*. Moreover, theoretically, Lindblad's spiral arms after unwinding in the sense of the rotation and reaching a reversing point, ultimately return to the vicinity of the original circular orbit, whilst in the present theory the arms after reversal wind round at increasing distance from the nucleus.

2. *Equations of motion referred to a fixed frame.*—The vector equation of motion, referred to a non-rotating frame attached to the nucleus of the nebula, is

$$\frac{d\boldsymbol{V}}{dt} = -\frac{\boldsymbol{P}-\boldsymbol{V}t}{t^2} - \frac{c^3tM}{M_0}\frac{\boldsymbol{P}}{|\boldsymbol{P}|^3}, \tag{5}$$

corresponding to the polar equations

$$\ddot{r} - r\dot{\theta}^2 = -\frac{r-t\dot{r}}{t^2} - \frac{c^3tM}{M_0}\frac{1}{r^2} \tag{6}$$

and

$$r^2\dot{\theta} = \text{const.} \times t. \tag{7}$$

In the integrated equation of the orbits (1), (1′), put as a dimensionless parameter

$$q_0 = \frac{c^3t_0^3}{r_0^3}\frac{M}{M_0}, \tag{8}$$

so that the orbits are

$$\theta - \theta_0 = q_0^{\frac{1}{2}}\log(t/t_0), \quad r = r_0t/t_0, \tag{9), (9′}$$

whence

$$\dot{\theta} = q_0^{\frac{1}{2}}/t, \qquad r^2\dot{\theta} = q_0^{\frac{1}{2}}r_0^2t/t_0^2. \tag{10), (10′}$$

The orbits (9), (9′) satisfy (6) and (7).

3. *Use of a rotating frame of reference.*—Now, following Lindblad, adopt a rotating system of angular coordinates, rotating about C, the centre of the galaxy under consideration, with the variable angular velocity ω given by

$$\omega = \theta = q_0^{\frac{1}{2}}/t, \quad \dot{\omega} = -q_0^{\frac{1}{2}}/t^2. \tag{11), (11′}$$

By the usual transformation to a rotating frame we have

$$\boldsymbol{V} = \frac{d\boldsymbol{P}}{dt} = \frac{\partial\boldsymbol{P}}{\partial t} + \omega\boldsymbol{k}\wedge\boldsymbol{P}, \tag{12}$$

* B. Lindblad, *Arkiv. för Mat., Astron. och Fys.*, Bd. **20** A, No. 17, 1927; *Upsala Med.*, No. 26, 1927.

† W. M. Smart, *Stellar Dynamics*, Cambridge (1938), p. 397 ff.

where $\boldsymbol{k}$ is a unit vector normal to the plane of the galaxy, and

$$\frac{d\boldsymbol{V}}{dt}=\left(\frac{\partial}{\partial t}+\omega\boldsymbol{k}\wedge\right)\left(\frac{\partial\boldsymbol{P}}{\partial t}+\omega k\wedge\boldsymbol{P}\right)=\frac{\partial^2\boldsymbol{P}}{\partial t^2}+2\omega\boldsymbol{k}\wedge\frac{\partial\boldsymbol{P}}{\partial t}-\omega^2\boldsymbol{P}+\dot{\omega}\boldsymbol{k}\wedge\boldsymbol{P}. \quad (13)$$

The vector equation of motion (5) then becomes

$$\frac{\partial^2\boldsymbol{P}}{\partial t^2}+2\omega\boldsymbol{k}\wedge\frac{\partial\boldsymbol{P}}{\partial t}-\omega^2\boldsymbol{P}+\dot{\omega}\boldsymbol{k}\wedge\boldsymbol{P}=-\frac{1}{t^2}\left[\boldsymbol{P}-t\left(\frac{\partial\boldsymbol{P}}{\partial t}+\omega\boldsymbol{k}\wedge\boldsymbol{P}\right)\right]-\frac{c^3tM}{M_0}\frac{\boldsymbol{P}}{|\boldsymbol{P}|^3}. \quad (14)$$

In this, put

$$\boldsymbol{P}=x\boldsymbol{i}+y\boldsymbol{j}, \quad (15)$$

so that

$$\frac{\partial\boldsymbol{P}}{\partial t}=\dot{x}\boldsymbol{i}+\dot{y}\boldsymbol{j}, \qquad \frac{\partial^2\boldsymbol{P}}{\partial t^2}=\ddot{x}\boldsymbol{i}+\ddot{y}\boldsymbol{j},$$

$\boldsymbol{i}$, $\boldsymbol{j}$ being a pair of rotating unit vectors along the axes of the rotating coordinate system; $\boldsymbol{i}$, $\boldsymbol{j}$ behave as constants under the operator $\partial/\partial t$. The $\boldsymbol{i}$- and $\boldsymbol{j}$-components of the last equation of motion give respectively

$$\ddot{x}-2\omega\dot{y}-\omega^2x-\dot{\omega}y=-\frac{1}{t^2}[x-t\dot{x}+\omega ty]-\frac{Mc^3t}{M_0}\frac{x}{r^3}, \quad (16)$$

$$\ddot{y}+2\omega\dot{x}-\omega^2y+\dot{\omega}x=-\frac{1}{t^2}[y-t\dot{y}-\omega tx]-\frac{Mc^3t}{M_0}\frac{y}{r^3}, \quad (16')$$

where

$$r^2=x^2+y^2,$$

and ω and $\dot{\omega}$ are given by (11) and (11′).

4. It will readily be verified that (16) and (16′) are satisfied by the spiral orbit

$$x=r=r_0t/t_0, \qquad y=0.$$

5. *Varied motion.*—Next put

$$x=r_0t/t_0+\xi, \quad y=\eta, \quad (17), (17')$$

where ξ and η are small compared with r_0t/t_0. Then

$$\dot{x}=r_0/t_0+\dot{\xi}, \qquad \ddot{x}=\ddot{\xi}, \qquad \dot{y}=\dot{\eta}, \qquad \ddot{y}=\ddot{\eta},$$

$$\frac{x}{r^3}\sim\frac{1}{r_0^2}\left(\frac{t_0}{t}\right)^2\left(1-2\frac{\xi}{r_0}\frac{t_0}{t}\right),$$

and the equations of motion (16) and (16′) will be found to reduce approximately to

$$\ddot{\xi}-2q_0^{\frac{1}{3}}\frac{\dot{\eta}}{t}-3q_0\frac{\xi}{t^2}+2q_0^{\frac{1}{3}}\frac{\eta}{t^2}=-\frac{\xi-t\dot{\xi}}{t^2}, \quad (18)$$

$$\ddot{\eta}+2q_0^{\frac{1}{3}}\frac{\dot{\xi}}{t}-2q_0^{\frac{1}{3}}\frac{\xi}{t^2}=-\frac{\eta-t\dot{\eta}}{t^2}. \quad (18')$$

In these equations put

$$\xi=(t/t_0)X, \qquad \eta=(t/t_0)Y.$$

Then (18), (18′) reduce to

$$t^2\ddot{X}+t\dot{X}-2q_0^{\frac{1}{3}}t\dot{Y}-3q_0X=0, \quad (19)$$

$$t^2\ddot{Y}+t\dot{Y}+2q_0^{\frac{1}{3}}t\dot{X}=0. \quad (19')$$

Now transform the scale of time by writing

$$\tau = t_0 \log (t/t_0) + t_0, \tag{20}$$

so that

$$dt/t = d\tau/t_0. \tag{20'}$$

Then (19), (19′) become

$$t_0^2 \frac{d^2X}{d\tau^2} - 2q_0^{\frac{1}{2}} t_0 \frac{dY}{d\tau} - 3q_0 X = 0, \tag{21}$$

$$t_0^2 \frac{d^2Y}{d\tau^2} + 2q_0^{\frac{1}{2}} t_0 \frac{dX}{d\tau} = 0. \tag{21'}$$

It will be recognized that X, Y are the transforms of ξ, η to τ-measure.

Equation (21′) integrates as it stands to

$$t_0^2 dY/d\tau + 2q_0^{\frac{1}{2}} t_0 X = \text{const.}$$

As remarked by Lindblad and Smart, we can choose the star of reference, $(r_0, 0)$, so that the constant of integration is zero. We then have

$$t_0 dY/d\tau + 2q_0^{\frac{1}{2}} X = 0. \tag{22}$$

Substituting this in (21), we get

$$t_0^2 d^2X/d\tau^2 + q_0 X = 0. \tag{23}$$

Since $q_0 > 0$, it follows from this equation that the motion in X is restricted when τ increases indefinitely; and it then follows from (22) that the motion in Y is also restricted. This shows that the spiral orbits (1), (1′) are *stable* in the sense that deviations are restricted to a small fraction of the radius vector.

The solution of (23) and (22) is

$$X = a \cos\left(\frac{q_0^{\frac{1}{2}}}{t_0}\tau + \epsilon\right), \tag{24}$$

$$Y = -2a \sin\left(\frac{q_0^{\frac{1}{2}}}{t_0}\tau + \epsilon\right), \tag{24'}$$

whence

$$\xi = a \frac{t}{t_0} \cos\left(\frac{q_0^{\frac{1}{2}}}{t_0}\tau + \epsilon\right), \tag{25}$$

$$\eta = -2a \frac{t}{t_0} \sin\left(\frac{q_0^{\frac{1}{2}}}{t_0}\tau + \epsilon\right), \tag{25'}$$

where a is arbitrary. Hence in t-measure, the path in the rotating frame of reference is the expanding ellipse

$$\xi^2 + \tfrac{1}{4}\eta^2 = a^2 (t/t_0)^2. \tag{26}$$

Moreover, differentiating (25) and (25′) with regard to t, we have

$$\dot{\xi} = \frac{a}{t_0}\left[\cos\left(\frac{q_0^{\frac{1}{2}}}{t_0}\tau + \epsilon\right) - q_0^{\frac{1}{2}} \sin\left(\frac{q_0^{\frac{1}{2}}}{t_0}\tau + \epsilon\right)\right], \tag{27}$$

$$\dot{\eta} = -\frac{2a}{t_0}\left[\sin\left(\frac{q_0^{\frac{1}{2}}}{t_0}\tau + \epsilon\right) + q_0^{\frac{1}{2}} \cos\left(\frac{q_0}{t_0}\tau + \epsilon\right)\right], \tag{27'}$$

so that the velocity components $(\dot{\xi}, \dot{\eta})$ *in the rotating frame* satisfy the relation

$$\dot{\xi}^2 + \tfrac{1}{4}\dot{\eta}^2 = (a^2/t_0^2)(1 + q_0). \tag{28}$$

6. *Velocities relative to a non-rotating frame.*—As Lindblad originally pointed out, the velocities of the stars in our own galaxy relative to the local standard of rest are small compared with the orbital velocity in a strictly circular orbit, being of the order of 10–20 km. sec.$^{-1}$, as compared with the velocity in the circular orbit (at the Sun's distance from the centre of the galaxy) of the order of 300 km. sec.$^{-1}$. Hence, as in Lindblad's work, the foregoing analysis applies to motions in the vicinity of the Sun, in the case of our own galaxy. The relative velocities $\dot{\xi}$, $\dot{\eta}$ are however measured in a *rotating* frame, and it becomes necessary to translate them into velocities relative to axes *fixed in direction* but moving with the mean motion in the vicinity of the star of reference.

Following Lindblad and Smart, let (ξ, η) refer to a star in the close vicinity of the Sun, S_1, so that effectively the relative coordinates ξ, η of the star (but not of course the coordinate velocity components $\dot{\xi}$, $\dot{\eta}$) are those of the Sun. The polar coordinates of the star of reference S_0 (which is executing a strictly circular orbit in τ-measure) being (r_0, θ_0) at time t_0, or (r, θ) at any time t, the polar coordinates of S_1 may be taken to be (r_1, θ_1) where, if $\theta_1 - \theta_0 = \delta\theta$,

$$r + \xi = r_1, \tag{29}$$

$$\eta = r_1 \delta\theta, \tag{30}$$

to a sufficient order.*

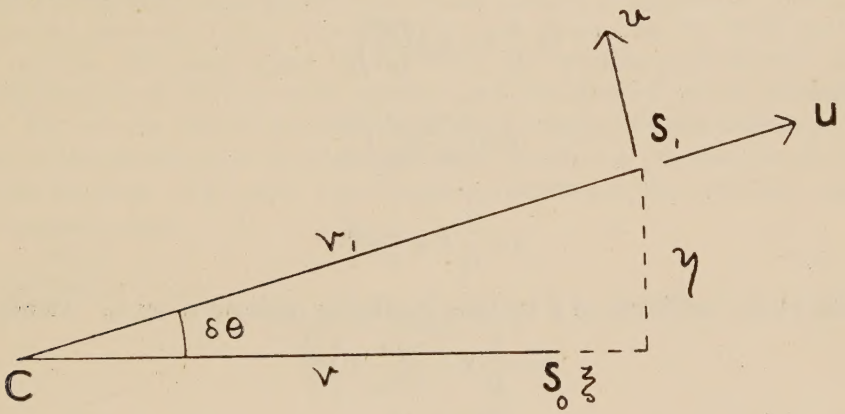

FIG. 1.

Let, now, (u, v) be the components of velocity of a star near S_1 referred to a *non-rotating frame* of reference anchored to the local standard of rest at S_1. Then, whilst the components of the star's motion at S_1, in the directions along and perpendicular to CS_0 (see Fig. 1) are

$$\dot{r} + \dot{\xi} - \eta\dot{\theta}, \qquad \dot{\eta} + (r + \xi)\dot{\theta},$$

the same components in terms of u and v are to the same approximation

$$\dot{r}_1 + u - r_1\dot{\theta}_1\delta\theta, \qquad v + r_1\dot{\theta}_1 + (\dot{r}_1 + u)\delta\theta.$$

Equating corresponding components and using (30), we get, first,

$$\dot{r} + \dot{\xi} - \eta\dot{\theta} = \dot{r}_1 + u - \eta\dot{\theta}_1.$$

* Smart supplies a calculation at this point, whilst Lindblad quotes results without proof. Their final results are correct, but Smart's analysis appears to me to be inconsistent in its approximations (see Section 12.16); he equates, in our notation, $\omega_1 r_1$ to ωr_1, but $\omega_1 r_1$ and ωr_1 differ by a quantity of the order of $\dot{\eta}$, which we are trying to find. It is in any case here necessary to refine on Smart's analysis, as we are taking into account the radial motion in the spiral orbits as well as the rotational motion.

But η is small, and $\dot{\theta}$ and $\dot{\theta}_1$ differ by a small quantity of the first order. Hence we have, to the desired order,

$$\dot{r}+\dot{\xi}=\dot{r}_1+u.$$

But $\dot{r}=r/t$, $\dot{r}_1=r_1/t$, and so

$$u=\dot{\xi}+(r-r_1)/t,$$

or, by (29),

$$u=\dot{\xi}-\xi/t,$$

or again, by the substitution preceding (19),

$$u=(t/t_0)\dot{X}=dX/d\tau. \tag{31}$$

Secondly, the equality of the components perpendicular to CS_0 gives

$$\dot{\eta}+(r+\xi)\dot{\theta}=v+r_1\dot{\theta}_1+\dot{r}_1\delta\theta,$$

whence, by (29),

$$v=\dot{\eta}-\frac{r_1}{t}\delta\theta+r_1(\dot{\theta}-\dot{\theta}_1),$$

or, using (30),

$$\begin{aligned} v&=\dot{\eta}-\frac{\eta}{t}+r_1(r-r_1)\left(\frac{d\dot{\theta}}{dr}\right)_1 \\ &=\frac{t}{t_0}\dot{Y}-r_1\xi\left(\frac{d\dot{\theta}}{dr}\right)_1. \end{aligned}$$

Now

$$\dot{\theta}=\left(\frac{c^3t_0^3}{r^3}\frac{M}{M_0}\right)^{\frac{1}{2}}\frac{1}{t}.$$

Hence

$$v=\frac{t}{t_0}\dot{Y}+\frac{3}{2}\xi\frac{q_0^{\frac{1}{2}}}{t},$$

whence in the coefficient of ξ we have justifiably replaced r_1 by r_0. Hence

$$\begin{aligned} v&=\frac{t}{t_0}\dot{Y}+\frac{3}{2}\left(\frac{t}{t_0}X\right)\frac{q_0^{\frac{1}{2}}}{t} \\ &=\frac{dY}{d\tau}+\frac{3}{2}\frac{q_0^{\frac{1}{2}}}{t_0}X. \end{aligned}$$

Using (22), we get

$$v=\frac{dY}{d\tau}-\frac{3}{4}\frac{dY}{d\tau}=\frac{1}{4}\frac{dY}{d\tau}. \tag{32}$$

Relations (31) and (32) relate the velocity components relative to the non-rotating local standard of rest at S_1 expressed in t-measure, to the velocity components in τ-measure, relative to the rotating frame of reference at S_0.

Now, by (24) and (24′),

$$\left(\frac{dX}{d\tau}\right)^2+\frac{1}{4}\left(\frac{dY}{d\tau}\right)^2=\frac{a^2q_0}{t_0^2}. \tag{33}$$

Introducing (31) and (32) in (33) we get

$$u^2+\tfrac{1}{4}(4v)^2=a^2q_0/t_0^2,$$

or

$$u^2+4v^2=a^2q_0/t_0^2. \tag{34}$$

7. *Star-streaming.*—Thus we establish, on our kinematic gravitational theory, the remarkable result that Lindblad obtained on the classical Newtonian theory of gravitation, that when we pass from a rotating to a non-rotating frame, the axes of the ellipse of velocities interchange themselves as regards their ratio. As remarked by Smart (p. 401), u, v, (which are his $\dot{\xi}_0$, $\dot{\eta}_0$) are the components of stellar velocities observed at S_1. When a is given, the components u, v referred to a non-rotating frame at S_1 satisfy (34), and for given a, and varying positions of S_0, there will be a family of orbits for each of which (34) will be satisfied.

The ellipse (26), or the corresponding ellipse satisfied by (X, Y), has its axis-major in the η-direction, i. e. *perpendicular to* CS_0. The velocity-ellipse (33) has its axes similarly oriented. But the velocity-ellipse (34) has its axis-major *parallel to* CS_1.

In each case the axis-major is just twice the axis-minor. The proof that when the velocity components satisfy (34) the *distribution* of velocities satisfies Schwarzschild's velocity-ellipse law (in two dimensions), is as given by Lindblad, or in more detail by Smart *, and I have nothing to add to it.

8. The observed ratio of the axes of the velocity-ellipse in our galaxy, in the direction of and perpendicular to the direction of the galactic centre, is not however 2 : 1. It is clear that whilst qualitatively the present analysis extends Lindblad's analysis of departures from circular motions to departures from spiral motions, the particular ratio 2 : 1 is due to our assumption that the whole gravitational motions take place under the influence of a massive point-nucleus at C. It has seemed worth while to work out this case in full detail to get the main ideas clear. But we now need to generalize our representation of spiral orbits and our account of star-streaming so as to take account of the effect of the mass-distribution round the nucleus. The paper which follows is devoted to this problem.

19 *Northmoor Road,*
Oxford :
1948 *June* 4.

* *Loc. cit.*, Section 12.17.

STAR-STREAMING AND THE STABILITY OF SPIRAL ORBITS IN SPIRAL NEBULAE

II.—MOTION IN AN EXTENDED DISTRIBUTION OF MATTER

E. A. Milne, F.R.S.

(Communicated by the Director of the University Observatory, Oxford)

(Received 1948 June 5)

Summary

The results of the previous paper are extended to the case where the gravitational field of the nebula is not merely that due to a point-nucleus. An appropriate expression is found for such a field, with the appropriate dependence on the time. Connection is made with well-known results of Lindblad and Oort.

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1. *Choice of a function to represent the attraction.*—When the gravitational acceleration under a central point-mass is expressed in τ-measure, its value is a function of the radius vector only. When the gravitational acceleration is partly due to material outside the nucleus, its value is still a function of the radius vector only provided that the orbits are circular or approximately circular. But if the orbits are spirals, the topographical distribution of the gravitating matter is changing with the time, and accordingly the gravitational acceleration will be a function of the time t as well as of the radius vector. This dependence on t is of course in addition to the implied dependence on t due to the proportionality of the "constant" of gravitation to t. This means that we cannot strictly follow Lindblad, Oort and Smart in taking a gravitational attraction represented by a function $K(r)$ of r only. (See e. g. Smart, *Stellar Dynamics*, Cambridge (1938), p. 394). Instead, in the present work we must replace

$$\gamma_t \frac{M}{r^2}, \qquad \text{or} \qquad \frac{c^3 t}{M_0}\frac{M}{r^2},$$

by an expression of the form

$$\gamma_t \frac{M}{r^2} F(r,t), \qquad \text{or} \qquad \frac{c^3 t}{M_0}\frac{M}{r^2} F(r,t).$$

Here $F(r,t)$ must be a function of zero physical dimensions, expressible as a function of arguments which must also be of zero physical dimensions. The only fundamental constant at our disposal being c, we shall satisfy this requirement by writing the attraction as a function of the single dimensionless variable r/ct, namely

$$\gamma_t \frac{M}{r^2} F\left(\frac{r}{ct}\right), \qquad \text{or} \qquad \frac{c^3 t}{M_0}\frac{M}{r^2} F\left(\frac{r}{ct}\right), \tag{1}$$

in t-measure. In this form it is reminiscent of an old argument due to Eddington, namely that the length-measures must be compared, in a fundamental investigation, with the radius of the universe, which in the present theory is ct.

It may be enquired why we do not introduce an argument into F of the form r/r_1, where r_1 is the constant concerning the galactic nuclear region introduced

earlier. We do not of course exclude the possibility of the occurrence of r_1 in F, but we want the form of F to be such that when we transform to τ-measure, circular orbits (corresponding to the relation $r=r_0(t/t_0)$ in t-measure) must enclose a constant mass, and hence be described under a constant gravitational attraction. This condition is satisfied only if r occurs in F in the form r/t, where of course t is measured from the natural origin of time.

There is a further condition which F must satisfy. When the corresponding spiral arms have been calculated as the locus of the present ($t=t_0$) positions of the particles in their orbits, they must be such that when continued along the actual orbits from time t_0 to time t, the new position of the spiral arm must be independent of t_0 and depend on t in the way in which the original (r_0, θ_0) equation of an arm depended on t_0. I pointed out previously* that this property was possessed by the equation to a spiral arm calculated for a massive point-nucleus, but it is not automatically possessed by the equation of an arm corresponding to an arbitrary attraction given by $F(r, t)$. It is, however, possessed by an attraction of the form (1), as we shall see.

2. *The orbits.*—Since the radial polar equation of motion in t-measure is now of the form

$$\ddot{r}-r\dot{\theta}^2=-\frac{r-t\dot{r}}{t^2}-\frac{c^3t}{M_0}\frac{M}{r^2}F\left(\frac{r}{ct}\right), \tag{2}$$

the orbits which are circular in τ-measure, namely those for which in t-measure we have

$$r=r_0(t/t_0), \tag{3}$$

require a motion in θ given by

$$-r_0\frac{t}{t_0}\left(\frac{d\theta}{dt}\right)^2=-\frac{c^3t}{M_0}\frac{M}{r_0^2}\frac{t_0^2}{t}F\left(\frac{r_0}{ct_0}\right),$$

or

$$\frac{d\theta}{dt}=\left[\frac{c^3t_0^3}{r_0^3}\frac{M}{M_0}F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}}\frac{1}{t}=\left[q_0F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}}\frac{1}{t}, \tag{4}$$

say. This integrates to give

$$\theta-\theta_0=\left[\frac{c^3t_0^3}{r_0^3}\frac{M}{M_0}F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}}\log\frac{t}{t_0}. \tag{5}$$

Thus the (r, θ) locus of an orbit is

$$\theta-\theta_0=\left[\frac{c^3t_0^3}{r_0^3}\frac{M}{M_0}F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}}\log\frac{r}{r_0}. \tag{6}$$

These are the same equiangular spirals as those described under the influence of a massive point-nucleus only, but are differently distributed with regard to the parameter r_0.

3. *Spiral arms.*—If a spiral arm is defined as the locus of the present positions of particles which have been emitted at various epochs t_1 in the past through a fixed point (r_1, β), then by (3) and (5) we require

$$\beta-\theta_0=\left[\frac{c^3t_0^3}{r_0^3}\frac{M}{M_0}F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}}\log\frac{t_1}{t_0}, \qquad r_1=r_0\frac{t_1}{t_0}, \tag{7}$$

* *Ap. J.*, **106**, 139, 1947.

whence eliminating t_1, the locus of present positions (r_0, θ_0) must satisfy the equation

$$\theta_0 = \beta + \left[\frac{c^3 t_0^3}{r_0^3}\frac{M}{M_0} F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}} \log\frac{r_0}{r_1}. \tag{8}$$

Such an arm depends as regards its detailed shape on the function $F(r_0/ct_0)$. It resembles however the arm for attraction by a massive point-nucleus only, in that the arm passes through (r_1, β), unwinds in the orbital direction to a maximum value of θ, then reverses and winds in the opposite direction, reaching $\theta = \beta$ again as $r_0 \to \infty$, provided the function $F(r_0/ct_0)$ does not increase too fast with increase of its argument.

Of course, if too large a fraction of the mass of the nebula is concentrated in a small number of discrete arms, the gravitational acceleration at greater distances will no longer possess rotational symmetry about the nucleus, and (1) will be no longer applicable. We can leave this case on one side for the present, and go on to develop the further consequences of (1).

4. *The permanent equation to a nebular arm.*—Since (8) represents a spiral arm at time $t = t_0$, the equation to the arm at time t will be obtained by eliminating r_0 and θ_0 from (8) by means of the parametric equations to the orbit, namely (3) and (5). We get

$$\theta - \left[\frac{c^3 t_0^3}{r_0^3}\frac{t^3}{t_0^3} F\left(\frac{r}{ct}\right)\right]^{\frac{1}{2}} \log\frac{t}{t_0} = \beta + \left[\frac{c^3 t_0^3}{r^3}\frac{t^3}{t_0^3} F\left(\frac{r}{ct}\right)\right]^{\frac{1}{2}} \log\left(\frac{r}{r_1}\frac{t_0}{t}\right),$$

which reduces to

$$\theta = \beta + \left[\frac{c^3 t^3}{r^3}\frac{M}{M_0} F\left(\frac{r}{ct}\right)\right]^{\frac{1}{2}} \log\frac{r}{r_1}. \tag{9}$$

But this is simply (8) with (r, θ) replacing (r_0, θ_0) and t replacing t_0. The remarkable thing about the elimination is that in the course of eliminating r_0 and θ_0 we have also eliminated t_0, and arrived at an equation of the same form in r, θ, t as (8) was in r_0, θ_0, t_0. This property could not have been guaranteed beforehand, as simple examples with other forms of $F(r, t)$ show that it is not automatically possessed by any arbitrary function $F(r, t)$.

Equation (9) accordingly represents the permanent form of a spiral arm, when formed under the attraction (1), at any epoch t.

5. *Star-streaming in the general gravitational field* (1).—We now generalize the work of the preceding paper. We consider the central region of the nebula, where discrete spiral arms have not properly differentiated themselves, and we proceed to consider the stability of the orbits given by (3) and (5).

Referred to rotating axes through the galactic nucleus, the Cartesian equations (16), (16′) of the preceding paper become now

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x - \dot{\omega} y = -\frac{1}{t^2}[x - t(\dot{x} - \omega y)] - \frac{c^3 t}{M_0}\frac{Mx}{r^3} F\left(\frac{r}{ct}\right), \tag{10}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y + \dot{\omega} x = -\frac{1}{t^2}[y - t(\dot{y} + \omega x)] - \frac{c^3 t}{M_0}\frac{My}{r^3} F\left(\frac{r}{ct}\right), \tag{10′}$$

with ω and $\dot{\omega}$ given by

$$\omega = \left[q_0 F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}} \frac{1}{t}, \qquad \dot{\omega} = -\left[q_0 F\left(\frac{r_0}{ct_0}\right)\right] \frac{1}{t^2}, \tag{11}$$

and

$$r^2 = x^2 + y^2.$$

As a verification, equations (10), (10′) should be satisfied by the solution

$$x = r_0(t/t_0), \qquad y = 0. \tag{12}$$

This is readily verified.

6. We now consider the disturbed motion

$$x = r_0\frac{t}{t_0} + \xi, \qquad y = \eta, \qquad r^2 = \left(r_0\frac{t}{t_0} + \xi\right)^2 + \eta^2. \tag{13}$$

The attraction term on the right of (10) approximates to

$$-\frac{c^3 t}{M_0}\frac{Mt_0^2}{r_0^2 t^2}\left(1 + \frac{\xi}{r_0}\frac{t_0}{t}\right)\left(1 - 3\frac{\xi}{r_0}\frac{t_0}{t}\right)\left[F\left(\frac{r_0}{ct_0}\right) + \frac{\xi}{ct}F'\left(\frac{r_0}{ct_0}\right)\right],$$

when ξ and η are small compared with $r_0(t/t_0)$. This in turn reduces to

$$-q_0 F\left(\frac{r_0}{ct_0}\right)\frac{r_0}{tt_0}\left[1 - 2\frac{\xi}{r_0}\frac{t_0}{t} + \frac{\xi}{ct}\frac{F'(r_0/ct_0)}{F(r_0/ct_0)}\right].$$

Similarly, the attraction term on the right of (10′) reduces, to a sufficient order, to

$$-q_0 F\left(\frac{r_0}{ct_0}\right)\frac{\eta}{t^2}.$$

Using these, we find that (10) and (10′) reduce to

$$\ddot{\xi} - 2\left[q_0 F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}}\left(\frac{\dot{\eta}}{t} - \frac{\eta}{t^2}\right) - q_0\frac{\xi}{t^2}\left[3F\left(\frac{r_0}{ct_0}\right) - \frac{r_0}{ct_0}F'\left(\frac{r_0}{ct_0}\right)\right] = -\frac{\xi - t\dot{\xi}}{t^2}, \tag{14}$$

and

$$\ddot{\eta} + 2\left[q_0 F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}}\left(\frac{\dot{\xi}}{t} - \frac{\xi}{t^2}\right) = -\frac{\eta - t\dot{\eta}}{t^2}. \tag{14′}$$

In these put

$$\xi = (t/t_0)X, \qquad \eta = (t/t_0)Y. \tag{15), (15′}$$

We find

$$t^2\ddot{X} + t\dot{X} - 2\left[q_0 F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}} t\dot{Y} - q_0 F\left(\frac{r_0}{ct_0}\right)\left[3 - \frac{r_0}{ct_0}\frac{F'}{F}\right]X = 0, \tag{16}$$

$$t^2\ddot{Y} + t\dot{Y} + 2\left[q_0 F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}} t\dot{X} = 0. \tag{16′}$$

Now transform the scale of time from t to τ, where

$$\tau = t_0 \log(t/t_0) + t_0, \qquad dt/t = d\tau/t_0. \tag{17}$$

Then (16) and (16′) become

$$t_0^2\frac{d^2X}{d\tau^2} - 2\left[q_0 F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}} t_0\frac{dY}{d\tau} - q_0 F\left(\frac{r_0}{ct_0}\right)\left[3 - \frac{r_0}{ct_0}\frac{F'}{F}\right]X = 0, \tag{18}$$

and

$$t_0^2\frac{d^2Y}{d\tau^2} + 2\left[q_0 F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}} t_0\frac{dX}{d\tau} = 0. \tag{18′}$$

7. *Stability of the spiral orbits.*—The last equation integrates to give

$$t_0\frac{dY}{d\tau}+2\left[q_0F\left(\frac{r_0}{ct_0}\right)\right]^{\frac{1}{2}}X=\text{const.}=0, \tag{19}$$

say, by choice of the star of reference S_0. Substituting for $dY/d\tau$ from (19) in (18) we get

$$t_0^2\frac{d^2X}{d\tau^2}+q_0F\left(\frac{r_0}{ct_0}\right)\left[1+\frac{r_0}{ct_0}\frac{F'(r_0/ct_0)}{F(r_0/ct_0)}\right]X=0. \tag{20}$$

This is a simple harmonic equation for X, and we get stability with respect to X provided that

$$F\left(\frac{r_0}{ct_0}\right)+\frac{r_0}{ct_0}F'\left(\frac{r_0}{ct_0}\right)>0. \tag{21}$$

If $F\propto r^{-n}$, this condition requires $n<1$. Hence the gravitational attraction, which is proportional to F/r^2 or to r^{-2-n}, must decrease outwards more slowly than the inverse cube of the distance, as on the classical theory of gravitation.

Assuming (21) is satisfied, put

$$q_0\left[F\left(\frac{r_0}{ct_0}\right)+\frac{r_0}{ct_0}F'\left(\frac{r_0}{ct_0}\right)\right]=Q_0. \tag{22}$$

Then

$$X=a\cos\left(\frac{Q_0^{\frac{1}{2}}}{t_0}\tau+\epsilon\right), \tag{23}$$

and from (19),

$$Y=-2a\left[\frac{F(r_0/ct_0)}{F(r_0/ct_0)+(r_0/ct_0)F'(r_0/ct_0)}\right]^{\frac{1}{2}}\sin\left(\frac{Q_0^{\frac{1}{2}}}{t_0}\tau+\epsilon\right). \tag{23'}$$

Hence the (X, Y) locus in the rotating frame is the ellipse

$$X^2+\tfrac{1}{4}Y^2\left[1+\frac{r_0}{ct_0}\frac{F'(r_0/ct_0)}{F(r_0/ct_0)}\right]=a^2 \tag{24}$$

The locus in (ξ,η) is the expanding ellipse

$$\xi^2+\tfrac{1}{4}\eta^2\left[1+\frac{r_0}{ct_0}\frac{F'(r_0/ct_0)}{F(r_0/ct_0)}\right]=a^2\frac{t^2}{t_0^2} \tag{25}$$

Further, the velocity components $(\dot\xi,\dot\eta)$ satisfy the equation

$$\dot\xi^2+\tfrac{1}{4}\dot\eta^2\left[1+\frac{r_0}{ct_0}\frac{F'(r_0/ct_0)}{F(r_0/ct_0)}\right]=\frac{a^2}{t_0^2}(1+Q_0). \tag{26}$$

8. *Expressions in terms of Oort's constants A and B.*—We wish to express F and its derivative in terms of Oort's constants A and B, which express the relative radial velocities and transverse velocities as trigonometrical functions of the galactic longitude. We use of course our special orbital motion in a spiral, given by (3) and (5). The velocity components in the orbit, in the usual notation, are

$$\Pi(r,\theta)=\dot r=r_0/t_0=r/t,$$

$$\Theta(r,\theta)=r\dot\theta=r\left[qF\left(\frac{r}{ct}\right)\right]^{\frac{1}{2}}\frac{1}{t},$$

where

$$q=\frac{c^3t_0^3}{r^3}\frac{M}{M_0}.$$

We have the usual diagram (see Fig. 1). Then if P is a star near S, in the galactic plane, at distance s from S and of galactic longitude λ measured from

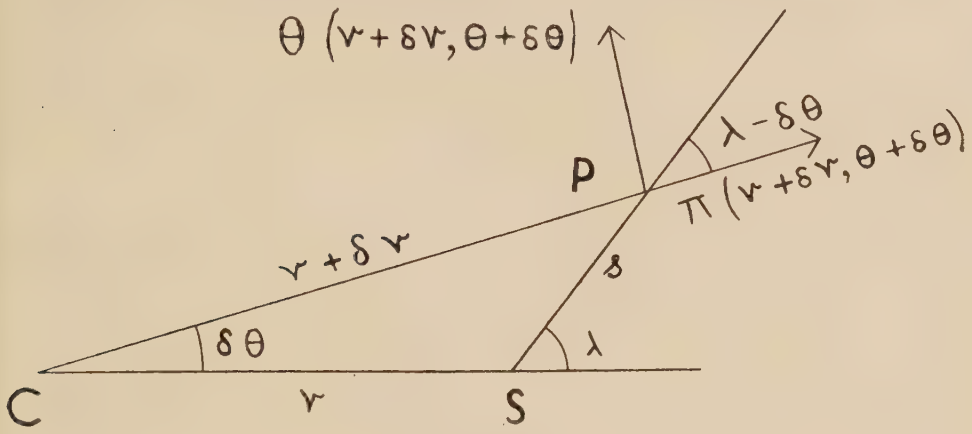

FIG. 1.

the direction away from the galactic centre C, we have approximately, from the triangle CSP,

$$\frac{s}{\delta\theta}=\frac{r}{\sin(\lambda-\delta\theta)}=\frac{r+\delta r}{\sin\lambda}=\frac{\delta r}{\delta\theta\cos\lambda},$$

so that, sufficiently accurately,

$$\delta r=s\cos\lambda,\qquad \delta\theta=(s\sin\lambda)/r.$$

The relative radial velocity, viewed from S, is

$$\Pi(r+\delta r,\theta+\delta\theta)\cos(\lambda-\delta\theta)+\Theta(r+\delta r,\theta+\delta\theta)\sin(\lambda-\delta\theta)$$
$$-\Pi(r,\theta)\cos\lambda-\Theta(r,\theta)\sin\lambda,$$

which will be found, with the above values of Π and Θ, to reduce to

$$s/t+sA\sin 2\lambda, \tag{27}$$

with A given by

$$A=\frac{1}{2t}\left[\frac{d}{dr}r(qF)^{\frac{1}{2}}-(qF)^{\frac{1}{2}}\right]=\frac{1}{2}\frac{r}{t}\frac{d}{dr}(qF)^{\frac{1}{2}}. \tag{28}$$

The transverse relative velocity is

$$\Theta(r+\delta r,\theta+\delta\theta)\cos(\lambda-\delta\theta)-\Pi(r+\delta r,\theta+\delta\theta)\sin(\lambda-\delta\theta)$$
$$-\Theta(r,\theta)\cos\lambda+\Pi(r,\theta)\sin\lambda,$$

which will be found to reduce to

$$sB+sA\cos 2\lambda, \tag{27'}$$

with B given by

$$B=\frac{1}{2t}\left[\frac{d}{dr}r(qF)^{\frac{1}{2}}+(qF)^{\frac{1}{2}}\right]. \tag{28'}$$

Thus

$$B-A=(qF)^{\frac{1}{2}}/t=\dot{\theta}, \tag{29}$$

the usual relation. Formulae (27) and (27′) are Oort's formulae, adapted to the presence of the radial expansion.

9. *Reduction of velocity components of star-streaming to a non-rotating frame.*—We now have to generalize the calculation of Section 6, Paper I. The relation

$$u=dX/d\tau \tag{30}$$

may be taken as it stands, (u, v) being as before the components of velocity of a star near S_1 in a non-rotating frame moving with S_1. For the transverse component, we take over the relation

$$v=\frac{t}{t_0}\dot{Y}-r_1\xi\left(\frac{d\dot{\theta}}{dr}\right)_1,$$

but now

$$\dot{\theta}=\left[\frac{c^3t_0^3}{r^3}\frac{M}{M_0}F\left(\frac{r}{ct_0}\right)\right]_1^{\frac{1}{2}}\frac{1}{t}.$$

Hence

$$v=\frac{t}{t_0}\dot{Y}-r_0\left(\frac{t}{t_0}X\right)\frac{1}{t}\left[\frac{d}{dr}(qF)^{\frac{1}{2}}\right]_0,$$

where we have now justifiably replaced r_1 by r_0.

Using (19), this gives

$$v=\frac{dY}{d\tau}\left[1+\frac{r_0}{2(q_0F)^{\frac{1}{2}}}\frac{d}{dr_0}(q_0F)^{\frac{1}{2}}\right]. \tag{31}$$

Here, since $q_0\propto r_0^{-3}$, the square bracket may be written

$$\begin{aligned}\left[1+\tfrac{1}{2}r\frac{d}{dr}(\tfrac{1}{2}\log qF)\right]_{r=r_0}&=1+\tfrac{1}{4}\left[-3+\frac{r_0}{ct_0}\frac{F'(r_0/ct_0)}{F(r_0/ct_0)}\right]\\&=\tfrac{1}{4}\left[1+\frac{r_0}{ct_0}\frac{F'(r_0/ct_0)}{F(r_0/ct_0)}\right].\end{aligned} \tag{32}$$

Now from (23) and (23′), the velocity components in the rotating frame, $dX/d\tau$ and $dY/d\tau$, in τ-measure, satisfy

$$\left(\frac{dX}{d\tau}\right)^2+\frac{1}{4}\frac{F(r_0/ct_0)+(r_0/ct_0)F'(r_0/ct_0)}{F(r_0/ct_0)}\left(\frac{dY}{d\tau}\right)^2=\frac{a^2Q_0}{t_0^2}. \tag{33}$$

Hence by (30) and (31), making use of (32), u and v satisfy

$$u^2+4\frac{F(r_0/ct_0)}{F(r_0/ct_0)+(r_0/ct_0)F'(r_0/ct_0)}v^2=\frac{a^2Q_0}{t_0^2}. \tag{34}$$

Thus, just as for motion about a massive point-nucleus only, we have the result that, when we pass from the velocity components (in τ-measure) in the rotating frame to the velocity components (in t-measure) in a non-rotating frame, the axes of the ellipse of velocities interchange themselves, as shown first by Lindblad for circular motion on the classical Newtonian theory of gravitation.

10. *Expression of the ratio of the axes of the velocity ellipse in terms of Oort's "constants" A and B.*—The ratio of the axis of the velocity ellipse (34) in the

u-direction, i.e. in the direction of the galactic centre, to that in the v-direction, is

$$2\left[\frac{F(r_0/ct_0)}{F(r_0/ct_0)+(r_0/ct_0)F'(r_0/ct_0)}\right]^{\frac{1}{2}}, \tag{35}$$

which by the transformations leading to (32) is

$$\left[1+\frac{r_0}{2(q_0F)^{\frac{1}{2}}}\frac{d}{dr_0}(q_0F)^{\frac{1}{2}}\right]^{-\frac{1}{2}}, \tag{36}$$

which by (28) and (29) is

$$\left[1+\frac{A}{B-A}\right]^{-\frac{1}{2}}=\left[1-\frac{A}{B}\right]^{\frac{1}{2}}. \tag{37}$$

This formula is due to Lindblad, and we thus establish its extended validity to the case of spiral orbits. If we use, as before, ω to denote the value of $\dot{\theta}$ in the vicinity considered, the ratio of axes (37) may also be written in the form

$$\left[\frac{\omega}{\omega+A}\right]^{\frac{1}{2}}, \tag{37'}$$

since $B=A+\omega$, in agreement with the formula given by Smart (Section 12.17, p. 402).

It must be remembered that ω is negative, and numerically greater than A, for the vicinity of the Sun. Thus the ratio (37′) is greater than unity, in accordance with the known result that the longer axis of the Schwarzschild velocity-ellipse is in the galactic plane and approximately in the direction of the galactic centre. The numerical value is obtained by substituting the values

$$A=0{\cdot}018, \qquad B=-0{\cdot}013,$$

in units of kms. sec.$^{-1}$ parsec^{-1}, which gives for the ratio of the axes

$$\sqrt{0{\cdot}031/0{\cdot}013}=1{\cdot}55,$$

which (as shown by Lindblad and Smart) is in agreement with the results of observations of stellar velocities.

11. We have thus shown that the known results in the theory of star-streaming hold good also on the kinematic theory of gravitation, for the spiral orbits which replace the circular orbits of the classical theory of gravitation; and the appropriate connections between the formulae on the two theories have been worked out in detail.

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1948 *June* 4.

INWARD INTEGRATION OF THE STELLAR EQUATIONS

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(Received 1948 June 4)

Summary

A method of inward integration of the equations of stellar structure is applied and extended to cover the cases of radiation pressure and electron opacity.

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1. *Introduction.*—A paper by Hoyle * gives a method of inward integration of the equations of stellar structure. It is shown there that for a region in which the transfer of energy is by radiation, the integration of the equations can, with a certain degree of approximation, be reduced to a one parameter problem. The purpose of this paper is to show how this method can be applied and to indicate the results that it gives for a radiative envelope fitted to a convective core. Comparison with the numerical results obtained by Cowling † and later by Hoyle and Lyttleton ‡ integrating outwards from the surface of the core can also be made. It is shown that the approximations do not lead to serious errors, that the integration of the equations is stable, and that the method is one which may effectively be used more generally. Thus, corresponding to definite assumptions as to composition and opacity, one is enabled to predict the luminosity and radius of a star of given mass. It should be noted that in general, when the radiation pressure is negligible, each solution of the equations corresponds to a one parameter series of possible configurations with the mass as parameter and that, therefore, even though the assumptions for composition and opacity may not be valid for a specified star it is highly probable that it would be valid for some set of stars.

2. We assume in the present paper that the composition is uniform throughout the star, with mean molecular weight μ. We first take the case actually considered by Hoyle §, namely a convective core with a radiative envelope in which the opacity is given by Kramers' law, radiation pressure being neglected in both the core and the envelope. The Kramers' law does not give the opacity due to photoelectric effect with complete accuracy, as the guillotine factor is neglected, but it does give the main variation, and the accuracy it gives is sufficient at this stage.

In Hoyle's paper, a numerical constant Θ_2 is introduced, the significance of which will appear later. A standard radius r_2 is defined in terms of the radius of

* F. Hoyle, *M.N.*, **105**, 23, 1945.
† T. G. Cowling, *M.N.*, **96**, 42, 1935.
‡ F. Hoyle and R. A. Lyttleton, *M.N.*, **102**, 177, 1942.
§ *Loc. cit.*

the star r_s by

$$r_2 = \frac{\Theta_2 - 1}{\Theta_2} r_s.$$

The crucial assumption is now that with a suitable choice of Θ_2, the mass outside r_2 can be neglected. This implies * that for $r > r_2$ (in the notation of Hoyle's paper)

$$\bar{T} = \bar{T}_2 \left[\frac{\Theta_2}{z} - (\Theta_2 - 1) \right], \qquad \bar{\rho} = \bar{\rho}_2 \left(\frac{\bar{T}}{\bar{T}_2} \right)^{3\cdot25},$$

where $z = r/r_2$ and $\bar{\rho}_2$, $\bar{T}_2$ are the values of $\bar{\rho}$ and $\bar{T}$ respectively at r_2.

In the radiative region $r \leqslant r_2$, the following equations then apply

$$\frac{d(\rho T)}{dz} = -4\cdot25\Theta_2 \frac{\rho M}{z^2}, \tag{1}$$

$$\frac{dM}{dz} = \phi \rho z^2, \tag{2}$$

$$\frac{dT}{dz} = -\Theta_2 \frac{\rho^2}{T^{6\cdot5}} \frac{1}{z^2}, \tag{3}$$

where $T = \bar{T}/\bar{T}_2$, $\rho = \bar{\rho}/\bar{\rho}_2$ and $M = \bar{M}/\bar{M}_2 = \bar{M}/\bar{M}_s$. Here ϕ is defined in terms of the variables at r_2 by

$$\phi = \frac{4\pi r_2^3 \bar{\rho}_2}{\bar{M}_s}, \tag{4}$$

while it follows from Kramers' opacity formula, in the customary notation, that

$$\Theta_2 = \frac{3\bar{\alpha}}{16\pi} \frac{(1 - X^2)L}{acr_2} \frac{\bar{\rho}_2^2}{\bar{T}_2^{7\cdot5}}. \tag{5}$$

An additional relation is given by Bethe's law. As is shown by Bethe †, the energy generation depends so critically on temperature that throughout the radiation region $L(r) = L$. Then

$$L = 4\pi E \int r^2 \bar{\rho}^2 \bar{T}^{\eta} dr \tag{6}$$

taken throughout the convective core.

We define n by the equation $n + 1 = d(\log \bar{P})/d(\log \bar{T})$. Then

$$(n+1)\Theta_2 = \frac{G\mu}{\mathscr{R}} \frac{\bar{M}\bar{T}^{6\cdot5}}{\bar{\rho}^2} \frac{\bar{\rho}_2^2}{r_2 \bar{T}_2^{7\cdot5}}, \tag{7}$$

and, since $n = 3\cdot25$ for $r \geqslant r_2$,

$$4\cdot25\Theta_2 = \frac{G\mu}{\mathscr{R}} \frac{\bar{M}_s}{r_2 \bar{T}_2}. \tag{8}$$

Hence

$$n + 1 = 4\cdot25 M T^{6\cdot5} \rho^{-2}. \tag{9}$$

The actual integration of equations (1), (2) and (3) was carried out by means of the differential analyser of the Cambridge University Mathematical Laboratory and it was found convenient to express the equations in terms of T and the variables s, u, W defined by $z = 1/s$, $\rho = uT^{3\cdot25}$, $M = WT$. s increases from 1 at r_2 to the value s_1 at the surface of the core.

* *Cf.* Chandrasekhar, *An Introduction to the Theory of Stellar Structure*, Chicago (1938), p. 300, equations (66) and (67).

† H. A. Bethe, *Phys. Rev.*, **55**, 452, 1939.

The equations (1), (2), (3) then become

$$T\frac{du}{ds}+4{\cdot}25u\frac{dT}{ds}=4{\cdot}25\Theta_2 uWT, \tag{10}$$

$$\frac{dW}{ds}+W\frac{d(\log T)}{ds}=-\frac{\phi uT^{2{\cdot}25}}{s^4}, \tag{11}$$

$$\frac{dT}{ds}=\Theta_2 u^2, \tag{12}$$

or, in the form in which the differential analyser actually performed the integration,

$$u=\Theta_2\left[2\int Wd\int u\,ds-\int u\,d(\log T)\right],$$

$$W=-\phi\int T^{2{\cdot}25}s^{-4}d\int u\,ds-\int Wd(\log T),$$

$$T=\Theta_2\int u\,d\int u\,ds,$$

with $u=T=W=1$ at $s=1$.

The integrations were performed with $\Theta_2=2$. It is clear that less error would be involved if Θ_2 were >2, but taking $\Theta_2=3$, the required ranges of the variables s and T become considerably larger and it was found that the increase in ranges made the use of the differential analyser less appropriate.

The conditions at the surface of the core are that the mass, pressure, temperature and temperature gradient should be continuous.

Thus $(n+1)$ must be continuous. In the convective core $n=1{\cdot}5$. This implies that the fit will occur when $s=s_1$ where s_1 is defined as the point at which the variables take the values u_1, T_1, W_1 satisfying $2{\cdot}5=4{\cdot}25W_1T_1u_1^{-2}$ from (9).

The procedure is therefore to take a series of values of ϕ and for each value find the corresponding s_1. The correct value of ϕ will be that for which the temperature gradient at s_1 is continuous.

To find the temperature gradient at s_1 for the radiative region in terms of the Emden units, in which the solutions for the convective core are tabulated *, it is necessary to find the radial coordinate x_1 in Emden units, corresponding to s_1. If y_1 is the Emden temperature at x_1, we have $\frac{\bar{T}_2}{\bar{T}_0}T_1=y_1$ and $\frac{\bar{\rho}_2}{\rho_0}\rho_1=y_1^3$ where the suffix 0 refers to the central values. Hence, since

$$s_1x_1=\frac{r_2}{[5\mathscr{R}\bar{T}_0/8\pi G\mu\bar{\rho}_0]^{\frac{1}{2}}},$$

and from (4) and (8)

$$r_2^2=\frac{8{\cdot}5\phi\mathscr{R}\bar{T}_2}{4\pi G\mu\bar{\rho}_2},$$

we have

$$s_1x_1=\left(\frac{17\phi}{5}u_1T_1^{2{\cdot}25}y_1^{-0{\cdot}5}\right)^{\frac{1}{4}}. \tag{13}$$

x_1 is found from this by successive approximation.

The temperature gradient of the radiative envelope at s_1 in the Emden units is

$$-2[\rho_1^2T_1^{-6{\cdot}5}z_1^{-2}]\frac{\bar{T}_2}{\bar{T}_0}\frac{z_1}{x_1}=-\left(\frac{5y_1^5}{4{\cdot}25\phi}\right)^{\frac{1}{4}}s_1^2u_1^{\frac{3}{4}}T_1^{-\frac{1}{4}}. \tag{14}$$

* Cf. *British Association Mathematical Tables*, Vol. II, 1932.

This has then to be compared with the y' of the Emden solutions.

The results obtained were as follows:—

ϕ	s_1	T_1	W_1	u_1	x_1	Temperature Gradient (at s_1, Emden)	Temperature Gradient (at s_1, radiative region)
0·40	3·138	3·56	0·0493	0·546	1·220	−0·3263	−0·3349
0·405	3·072	3·51	0·0512	0·553	1·247	−0·3302	−0·3303
0·41	3·016	3·47	0·0530	0·559	1·271	−0·3336	−0·3254
0·42	2·945	3·40	0·0552	0·565	1·298	−0·3372	−0·3166
0·43	2·853	3·32	0·0580	0·572	1·334	−0·3418	−0·3122
0·44	2·798	3·26	0·0601	0·577	1·357	−0·3441	−0·3081
0·45	2·722	3·19	0·0624	0·582	1·388	−0·3481	−0·3006
0·50	2·473	2·94	0·0722	0·599	1·514	−0·3601	−0·2694
0·55	2·288	2·74	0·0812	0·617	1·637	−0·3682	−0·2450

Thus the required value of ϕ is 0·405.

The values of u, T for a number of values of s for $\phi = 0{\cdot}405$ are given below:

s	T	u	s	T	u
1	1	1	2·0	2·60	0·77
1·2	1·39	0·98	2·5	3·10	0·66
1·4	1·74	0·93	3·0	3·47	0·57
1·6	2·07	0·88	3·07	3·51	0·55
1·8	2·35	0·825			

The mass-luminosity relation resulting from these figures is considered in Section 5.

3. Similar integrations have also been carried out for the case in which the opacity is entirely due to electron scattering. That is, the opacity is equal to $\kappa(1+X)$, where κ is a constant which in c.g.s. units is equal to 0·19.

The equation corresponding to (5) of Hoyle's paper is therefore

$$\tfrac{1}{3}ac\frac{d\bar{T}^4}{dr} = -\frac{\kappa(1+X)\bar{\rho}L}{4\pi r^2}.$$

In this case $\bar{\rho}/\bar{T}^3$ tends to a limit at the surface and r_2 is defined such that $\bar{\rho}$ is proportional to $\bar{T}^3$ for $r > r_2$. $\bar{T}_2$, $\bar{\rho}_2$ are defined as before and $\bar{M}(r_2) = \bar{M}_s$.

If we now put

$$\phi = \frac{4\pi r_2^3 \bar{\rho}_2}{\bar{M}_2}, \tag{15}$$

$$\Theta_2 = \frac{3\kappa(1+X)L}{16\pi ac}\frac{\bar{\rho}_2}{r_2\bar{T}_2^4}, \tag{16}$$

we get

$$(n+1)\Theta_2 = \frac{G\mu}{\mathscr{R}}\frac{\bar{M}\bar{T}^3}{\bar{\rho}}\frac{\bar{\rho}_2}{r_2\bar{T}_2^4}, \qquad r_2 = \frac{\Theta_2 - 1}{\Theta_2}r_s, \tag{17}$$

and, since $n = 3$ when $r = r_2$,

$$4\Theta_2 = \frac{G\mu}{\mathscr{R}}\frac{\bar{M}_s}{r_2\bar{T}_2}. \tag{18}$$

Defining z, ρ, T, M as before and putting $\rho = uT^3$, $z = 1/s$, the equations become

$$Tdu/ds + 4u\,dT/ds = 8uM, \tag{19}$$

$$dM/ds = -\phi u T^3, \tag{20}$$

$$dT/ds = 2u, \tag{21}$$

for the case $\Theta_2 = 2$.

These were solved by the differential analyser in the form

$$u = 4\int (M-u)d(\log T),$$

$$M = -\tfrac{1}{2}\phi \int T^3/s^4\,dT,$$

$$T = 2\int u\,ds.$$

As before, the integrations were carried out for various values of ϕ and in this case the fit on to the convective core occurred at the point s_1, where M_1, u_1 satisfy the equation $2{\cdot}5 = 4M_1/u_1$.

Expressing as before the equality of the corresponding radial distance, we have

$$s_1 x_1 = \left(\frac{16\phi}{5}\, u_1 T_1^2 y_1^{-0{\cdot}5}\right)^{\frac{1}{2}}. \tag{22}$$

To express the continuity of dT_1/dr, we have to equate

$$\left(\frac{5u_1}{4\phi}\right)^{\frac{1}{2}} y_1^{\frac{4}{5}} \frac{s_1^2}{T_1^2}$$

to the value of the temperature gradient given by the Emden tables at x_1.

The results obtained were as follows:—

ϕ	s_1	T_1	M_1	u_1	x_1	$-$Temperature gradient (at s_1, Emden)	$-$Temperature gradient (at s_1, radiative region)
0·5	2·67	3·39	0·293	0·469	1·165	0·318	0·504
0·6	2·21	2·84	0·323	0·519	1·396	0·349	0·415
0·68	1·99	2·55	0·344	0·551	1·564	0·364	0·359
0·7	1·95	2·50	0·348	0·557	1·599	0·366	0·350
0·8	1·79	2·28	0·371	0·593	1·806	0·374	0·293
0·9	1·66	2·10	0·390	0·622	2·028	0·372	0·235

The required value of ϕ is therefore about 0·67 and by interpolation we find that $s_1 = 2{\cdot}01$, $T_1 = 2{\cdot}58$, $M_1 = 0{\cdot}342$, $u_1 = 0{\cdot}548$.

The values of M, u, T for a number of values of s for $\phi = 0{\cdot}67$ are given below:

s	T	u	M
1	1	1	1
1·2	1·40	0·940	0·850
1·4	1·75	0·836	0·692
1·6	2·06	0·730	0·550
1·8	2·34	0·631	0·434
2·0	2·57	0·553	0·345
2·01	2·58	0·548	0·342

4. When radiation pressure is included, the equations are (using electron opacity)

$$\frac{d}{dr}\left(\frac{\mathscr{R}\bar{\rho}\bar{T}}{\mu\beta}\right) = -\frac{G\bar{\rho}\bar{M}}{r^2}, \tag{23}$$

$$\frac{d\bar{M}}{dr} = 4\pi r^2\bar{\rho}, \tag{24}$$

$$\frac{d\bar{T}}{dr} = -\frac{3\kappa(1+X)L}{16\pi acr^2}\frac{\bar{\rho}}{\bar{T}^3}. \tag{25}$$

These equations can be treated in a similar manner, provided that $d\beta/dr$ is neglected, where β is the ratio of gas pressure to total pressure. This does not in fact involve treating β as a constant but merely that one assumes that the term in which its derivative occurs is negligible.

With this approximation equation (23) becomes

$$\frac{d}{dr}\left(\frac{\mathscr{R}\bar{\rho}\bar{T}}{\mu}\right) = -\frac{G\bar{\rho}\bar{M}\beta}{r^2}. \tag{23'}$$

Here

$$\frac{1}{\beta} = 1 + \frac{\mu a}{3\mathscr{R}}\frac{\bar{T}^3}{\bar{\rho}}. \tag{26}$$

Defining r_2 as before and using the same variables T, u, M and y, the equations (23'), (24), (25) become

$$\frac{d}{dy}(uT^4) = 4\Theta_2(k+1)\frac{Mu^2T^3}{u+k}, \tag{27}$$

$$\frac{dM}{dy} = -\phi\frac{uT^3}{y^4}, \tag{28}$$

$$\frac{dT}{dy} = \Theta_2 u, \tag{29}$$

where

$$\phi = \frac{4\pi\bar{\rho}_2 r_2^3}{\bar{M}_s}, \tag{15}$$

$$\Theta_2 = \frac{3\kappa(1+X)\bar{\rho}_2 L}{16\pi acr_2\bar{T}_2^4} \tag{16}$$

and

$$k = \frac{1-\beta(r_2)}{\beta(r_2)}. \tag{30}$$

Putting $\Theta_2 = 2$, the equations (27), (28) and (29) were solved on the differential analyser in the form

$$u = 4\int\left\{\frac{(k+1)Mu}{u+k} - u\right\}d(\log T),$$

$$M = -\frac{\phi}{2}\int T^3 s^{-4}\,dT,$$

$$T = 2\int u\,ds,$$

with the initial conditions $u = M = T = 1$, $\beta = 1/(k+1)$ at $s = 1$.

In this case

$$(n+1)=d\frac{\left(\log\frac{\bar{\rho}\overline{T}}{\beta}\right)}{d(\log\overline{T})}=4(k+1)\frac{M\beta}{u},$$

and from (27) and (30),

$$\beta=u/(u+k). \tag{31}$$

At the surface of the core, β is continuous and the effective polytropic index is

$$\frac{3(8-7\beta_1)}{8-3\beta_1-3\beta_1^2},$$

where $\beta=\beta_1$ at $s=s_1$.

Thus the surface of the convective core occurs at the point $s=s_1$ at which

$$\frac{3(8-7\beta_1)}{8-3\beta_1-3\beta_1^2}+1=4(k+1)\frac{M_1\beta_1}{u_1},$$

or in terms of u, M only,

$$\frac{5u_1^2+40u_1k+32k^2}{2u_1^2+13u_1k+8k^2}=\frac{4M_1(k+1)}{u_1+k}=n_1+1 \quad \text{(say)}. \tag{32}$$

As before, this point s_1 is found for a series of values of ϕ. In the core, we assume that β is constant throughout. The justification of this is that in fact β varies like $\overline{T}^{3-n}$. For small radiation pressure, the radius of the core is small and so the variation of $\overline{T}$ through it is also small. For large radiation pressure, although the radius of the core is large, the value of n is very near to 3, so that the variation of $\overline{T}^{3-n}$ is again small.

The Emden radial coordinate x_1, corresponding to s_1, is thus given by

$$s_1x_1=\left(\frac{8\phi(k+1)}{n_1+1}\frac{u_1^2}{u_1+k}\frac{T_1^2}{y_1^{n_1-1}}\right)^{\frac{1}{2}}=u_1T_1\left(\frac{2\phi}{M_1y_1^{n_1-1}}\right)^{\frac{1}{2}}. \tag{33}$$

The temperature gradient at s_1 in the radiative region, when expressed in terms of Emden units is

$$-\frac{s_1^2}{T_1^2}\left(\frac{2M_1y_1^{n_1+1}}{\phi}\right)^{\frac{1}{2}}.$$

The correct value of ϕ is that which makes this expression equal to the y' of the Emden tables for the corresponding n_1.

This was carried out for $\beta(r_2)$ equal to 0·9, 0·8 and 0·5, i.e. for $k=\frac{1}{9}$, $\frac{1}{4}$ and 1. The solutions for intermediate values of $\beta(r_2)$ can clearly be found by interpolation.

The results obtained were as follows:—

$k=\frac{1}{9}$

ϕ	s_1	T_1	M_1	u_1	x_1	$-\left(\frac{dT}{dr}\right)$ Emden	$-\left(\frac{dT}{dr}\right)$ radiative
0·4	2·50	3·30	0·432	0·552	1·08	0·291	0·680
0·6	1·80	2·34	0·489	0·645	1·56	0·339	0·427
0·7	1·65	2·10	0·507	0·675	1·79	0·343	0·351
0·71	1·64	2·08	0·509	0·678	1·82	0·342	0·343
0·8	1·55	1·95	0·521	0·700	2·10	0·335	0·258

Thus the fit occurs for $\phi=0{\cdot}71$.

$k=\frac{1}{4}$

ϕ	s_1	T_1	M_1	u_1	x_1	$-\left(\frac{dT}{dr}\right)$ Emden	$-\left(\frac{dT}{dr}\right)$ radiative
0·4	2·01	2·71	0·557	0·660	1·21	0·303	0·641
0·5	1·74	2·24	0·582	0·706	1·41	0·321	0·568
0·6	1·59	2·04	0·595	0·730	1·70	0·331	0·424
0·7	1·49	1·88	0·610	0·755	1·98	0·325	0·326
0·75	1·45	1·81	0·617	0·770	2·24	0·310	0·252

Thus the fit occurs for $\phi=0{\cdot}70$.

$k=1$

ϕ	s_1	T_1	M_1	u_1	x_1	$-\left(\frac{dT}{dr}\right)$ Emden	$-\left(\frac{dT}{dr}\right)$ radiative
0·4	1·435	1·82	0·789	0·859	1·34	0·302	0·765
0·6	1·29	1·56	0·798	0·888	1·87	0·305	0·515
0·64	1·275	1·53	0·800	0·893	2·40	0·269	0·271

The required value of ϕ is thus 0·64.

5. We now consider the numerical values that our results give for the mass-luminosity relation and the relations holding between radius and mass. We also obtain expressions for the central density and temperature. These relations are in general dependent on Bethe's law of energy generation (6), which has not been introduced into the integration up to this point.

(6) may be written in the form

$$L=4\pi E\bar{\rho}_0^2\bar{T}_0^{\eta}\{\mathscr{R}(n_1+1)\bar{T}_0/4\pi G\mu\beta_1\bar{\rho}_0\}^{\frac{3}{2}}\int_0^{x_1}x^2\left\{\frac{\bar{T}(x)}{\bar{T}_0}\right\}^{\eta+2n_1}dx.$$

The integral is almost independent of n_1 and is equal to 0·07, approximately.

(i) For Kramers' opacity ($\beta_1=1$) using equations (4), (5) and (8),

$$L^{2\eta+5}=\left\{0{\cdot}28\pi\left(\frac{n_1+1}{17\Theta_2\pi}\right)^{\frac{3}{2}}E\frac{\rho_1^{\frac{1}{2}}T_1^{\eta+\frac{3}{2}}}{y_1^{\eta+\frac{1}{2}n+\frac{3}{2}}}\right\}^{-1}$$
$$\times\left(\frac{4\pi}{\phi}\right)^{4\eta+12{\cdot}5}\left(\frac{G\mu}{4{\cdot}25\Theta_2\mathscr{R}}\right)^{14\eta+45}\left(\frac{16\Theta_2\pi ac}{3\bar{\alpha}(1-X^2)}\right)^{2\eta+6}\bar{M}_s^{10\eta+31}.$$

Putting $\Theta_2=2$, $\phi=0{\cdot}405$, $n_1=1{\cdot}5$, $\eta=17{\cdot}25$ and inserting the values of ρ_1, T_1 and y_1, we have

$$L=[\overline{139}{\cdot}4939]\mu^{7{\cdot}25}(1-X^2)^{-1{\cdot}03}\bar{M}_s^{5{\cdot}15}. \qquad (34)$$

Using (4), (5) and (8) we get also

$$r_s=[\overline{290}{\cdot}7438]\mu^{15}(1-X^2)^{-2}L^{-2}\bar{M}_s^{11}, \qquad (35)$$

$$\bar{T}_2=[\overline{16}{\cdot}2783]\mu\bar{M}_s r_s^{-1}, \qquad (36)$$

$$\bar{\rho}_2=[\bar{1}{\cdot}4112]\bar{M}_s r_s^{-3}. \qquad (37)$$

Since

$$\frac{\bar{T}_2}{\bar{T}_0}T_1=y_1 \qquad \text{and} \qquad \frac{\bar{\rho}_2}{\bar{\rho}_0}\rho_1=y_1^{\frac{3}{2}},$$

(36) and (37) lead to

$$\bar{T}_0 = [\overline{16}\cdot 9380]\mu \bar{M}_s r_s^{-1}, \tag{38}$$

$$\bar{\rho}_0 = [0\cdot 9615]\bar{M}_s r_s^{-3}. \tag{39}$$

It is of some interest to note that the correct values of the luminosity and radius of the Sun are obtained only if $\mu = 0\cdot 807$ and $X = 0\cdot 967$. With this value of X and at the relevant temperatures, the Kramers' opacity law would certainly apply. If the conventional assumption that the material other than hydrogen is either helium or heavy elements ionized with mean molecular weight 2, these values of μ and X are inconsistent. However, if in fact the assumption of uniform composition is invalid, this result may be some indication of the mean molecular weights at the centre and in the outer regions, since X as used in our formulae refers throughout to its value in the radiative region, whereas μ refers also to the central regions, entering as it does into Bethe's law.

(ii) For electron opacity, we put $X = 1$ and therefore $\mu = \frac{1}{2}$. For any appreciably lower value of X, the electron opacity would not be predominant unless the temperature in the radiative region were of a higher order of magnitude than the theory indicates.

In this case L can be expressed as a function of $\bar{M}_s$ without using Bethe's law. In fact

$$L = \frac{\pi^2 ac}{96\kappa}\frac{1}{\phi(1+X)}\left(\frac{G}{\mathscr{R}}\right)^4 \beta_2^4 \bar{M}_s^3.$$

Using Bethe's law to give the radius, we have

$$L^{-\frac{\eta-1}{3}} = 0\cdot 28\pi E\left\{\frac{(n_1+1)\mathscr{R}}{4\pi G\mu\beta_1}\right\}^{\frac{1}{2}} \frac{u_1^{\frac{1}{2}} T_1^{\eta+3}}{y_1^{\eta+\frac{1}{2}n+\frac{3}{2}}}\left(\frac{2\mathscr{R}\phi}{\pi G\mu\beta_1}\right)^{\frac{1}{2}}\left(\frac{3\kappa\mathscr{R}\phi(1+X)}{16\pi^2 acG\mu\beta_1}\right)^{\frac{\eta+2}{3}}\frac{1}{r_2^{\eta+3}}.$$

Inserting numerical values, these two equations lead respectively to

$$L = [\overline{66}\cdot 2150]\frac{\beta_2^4 \bar{M}_s^3}{\phi}, \tag{40}$$

$$L^{-\frac{16\cdot 25}{3}} = [17\cdot 6316]\left(\frac{n_1+1}{\beta_1}\right)^{1\cdot 5}\left(\frac{\phi}{\beta_2}\right)^{\frac{41\cdot 5}{6}}\frac{u_1^{0\cdot 5} T_1^{20\cdot 25}}{y_1^{18\cdot 75+0\cdot 5n_1}} r_s^{-20\cdot 25}. \tag{41}$$

We have besides these the conditions expressed by equations (15), (16) and (18).

The approximation of zero radiation pressure is valid for all sufficiently small masses, i. e. equation (26) is approximately satisfied for all such masses, but for a given non-zero ratio of radiation pressure to gas pressure, equation (26) determines the mass uniquely.

Using also the equations $(\bar{T}_2/\bar{T}_0)T_1 = y_1$ and $(\bar{\rho}_2/\bar{\rho}_0)\rho_1 = y_1^{n_1}$, we have for $k = 0$ (when $n_1 = 1\cdot 5$)

$$L = [\overline{66}\cdot 3889]\bar{M}_s^3, \tag{42}$$

$$r_s = [\overline{17}\cdot 8652]\bar{M}_s^{0\cdot 80}, \tag{43}$$

$$\bar{T}_2 = [\overline{16}\cdot 0036]\bar{M}_s r_s^{-1}, \tag{44}$$

$$\bar{\rho}_2 = [\bar{1}\cdot 6300]\bar{M}_s r_s^{-3} \tag{45}$$

and

$$\bar{T}_0 = [0\cdot 5877]\bar{T}_2, \tag{46}$$

$$\bar{\rho}_0 = [1\cdot 2377]\bar{\rho}_2. \tag{47}$$

These are applicable for $\overline{M}_s < 4 \cdot 10^{34}$ g.

In particular for the Sun, $\overline{M}_s = [33{\cdot}2978]$ and

$$\begin{matrix} L = [34{\cdot}2823] \\ r_s = [10{\cdot}5856] \end{matrix} \quad \text{compared with observational values} \quad \begin{matrix} L = [33{\cdot}5775] \\ r_s = [10{\cdot}8421] \end{matrix}.$$

For $k \neq 0$, the results are given in the table below:

	$k=1$	$k=\frac{1}{4}$	$k=\frac{1}{9}$
$\overline{M}_s$	[35·6571]	[35·0693]	[34·8196]
L	[40·1759]	[39·1902]	[38·6394]
r_s	[12·2545]	[11·9127]	[11·8224]
$\overline{T}_c$	[7·6491]	[7·6067]	[7·5079]
$\bar{\rho}_c$	[$\bar{1}$·8633]	[0·2269]	[0·2319]

6. *Comparisons with previous integrations.*—The only numerical comparison that can be made with previous integrations are concerned with the Kramers' opacity. Interpolating in the Hoyle-Lyttleton solution * to get the point corresponding to r_2 for $\Theta_2 = 2$, ϕ turns out to be 0·556. Although the agreement between our value of ϕ and this one does not appear to be very good, it must be pointed out that for $\Theta_2 = 2{\cdot}35$, the Hoyle-Lyttleton solution gives $\phi = 0{\cdot}318$. The value of ϕ is thus very sensitive to changes in Θ_2 and hence is accurately determined only if the values of the variables at r_2 for specified Θ_2 are accurately determined. This is the weakest point in the connection between the methods of outward and inward integration. The agreement between the radius of the core (1·19 in the Cowling and Hoyle-Lyttleton solution and 1·25 in the present solution, expressed in Emden units) is extremely satisfactory in view of the degree of approximation used here. In any case, the method of inward integration is far less troubled by instabilities and is therefore much more straightforward to use.

In conclusion, the writer wishes to express her gratitude to Mr F. Hoyle for much helpful advice, and to Mr M. V. Wilkes, Director of the Cambridge University Mathematical Laboratory, for placing the facilities of the Laboratory at her disposal.

Newnham College,
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1948 *June* 3.

* F. Hoyle and R. A. Lyttleton, *M.N.*, **102**, 177, 1942.

ORBITAL ELEMENTS AND APSIDAL MOTION OF THE ECLIPSING BINARY, V 526 SAGITTARII

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(Received 1948 May 3; received in revised form 1948 July 22)

Summary

The light curve of V 526 Sagittarii, based on nearly 1300 photographic observations, is discussed. The orbital elements and the eccentricity of the orbit are derived. The period of apsidal revolution is determined and improved formulae are given for the prediction of the times of minima.

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1. *Introduction.*—The eclipsing binary, V 526 Sagittarii, was discovered by the writer on Riverview plates in 1934.* A preliminary light curve, published soon afterwards †, was based on only some 250 observations. It was possible, nevertheless, to show that the separation of the minima was changing and that the secondary minimum was much wider than the primary, and to deduce a rotation of the line of apsides. Photographic observations of the star have been continued at Riverview. Enough material is now available for a determination of the orbital elements and of the period of apsidal revolution within fairly narrow limits.

Using observations on Harvard plates covering forty years, Dugan and Miss Wright confirmed the apsidal motion.‡ They found that the residuals of the two minima from the mean period showed no sensible deviations from two intersecting straight lines. They gave linear formulae for the times of minima, which were adequate for purposes of prediction until a few years ago.

2. *Observations.*—Eight hundred and fifty-nine estimates, made by the writer on Lembang and Riverview plates, were available for the present study. The instrument and method used in making the estimates have been described in an earlier paper.§ During a short stay at Leiden Observatory in June, 1948, the writer made estimates of the variable on Franklin-Adams plates, taken at Johannesburg, centred on τ Sagittarii. The scale is much larger than that of the Riverview plates and the comparison stars are widely separated. However, by using a low-power magnifying glass with a large field, it was possible to view them all together, and satisfactory estimates could be made. The variable is, however, rather bright, and well above the optimum magnitude range for these plates.

The plates are distributed as follows:—

Plate series	Camera	Number of estimates	Interval covered
V	Zeiss Astrotessar, Lembang	37	J.D. 2424020 to 25133
X	Zeiss Astrotriplet, Lembang	98	26100 to 27364
R & P	Zeiss Astrotriplets, Riverview	511	27630 to 32494
G	Ross-Grubb-Parsons 4-in., Riverview	213	30134 to 32494
	Franklin-Adams, Leiden	407	25442 to 29165

* *A.N.*, **257**, 391, 1935.
† *Riverview Publ.*, No. 1, 11, 1935.
‡ R. S. Dugan and F. W. Wright, *Princeton Cont.*, No. 19, 7, 1939.
§ D. O'Connell, *Ann. Bosscha Sterrewacht* (Lembang), VIII, 1, 1935.

During several minima plates were taken at Riverview throughout the night, sometimes with the R and G cameras simultaneously. From two to six exposures were then made on each plate. Special precautions were taken when estimating these plates to keep the estimates independent of each other.

3. *Comparison Stars.*—The comparison stars are given in Table I, with their C.P.D. and H.D. numbers, spectra and photographic magnitudes. Stars *a*, *b*, *c*, *d* and *e* were also used by Miss Wright at Harvard. The magnitudes determined by her * have been used by the writer. The magnitudes of stars *a'* and *g* were interpolated from the Harvard sequence. The designation of the stars in the writer's earlier paper † is given in column 2. The magnitudes determined originally by the writer and given in that paper agree very closely in scale with the Harvard sequence. There is a small zero point difference of less than $0^m.2$, the Harvard magnitudes being fainter. All the estimates have been reduced with the new magnitudes.

TABLE I

Comparison Stars

Des.	Des. R.P. 1	C.P.D.	H.D.	Sp.	Pg. Mag.
		°			m
a	*d*	−30 5800	176720	Fo	9·24
a'		−31 5846	177020	A2	9·48
b	*e*	−31 5870	177506	Ao	9·84
c		−30 5846	177823	F8	9·94
d	*f*	−31 5866	177453	Ao	10·58
e		−31 5875			10·88
g	*g*	−31 5907			10·92

4. *Light Curve.*—All times were reduced to the Sun. For observations outside minima the phases were calculated with Dugan and Wright's mean period, $1^d.919407$. During primary minimum the period $1^d.919431$ was used, and during secondary minimum $1^d.919389$. After some preliminary trials these periods were found to agree best with the observed times of minima during the period covered by the Riverview observations. The period for secondary minimum, in particular, differs appreciably from Dugan and Wright's period, $1^d.919379$. This in itself is a sufficient indication that the residuals no longer follow the linear formulae obtained by them.

Only the X, R, P and G plates were used for the mean light curve. No significant systematic differences were found between estimates on plates of these series. The V plates are comparatively few in number and are appreciably earlier in date than the others. They may also differ somewhat from the others in colour-equation. The magnitude at maximum obtained from the Leiden plates is the same as that from the other plates, but the minima are shallower by more than $0^m.1$. This is probably due mainly to the difference in colour-equation between the Franklin-Adams lens and the others. The fact that the exposure of the Leiden plates (30 minutes) was much longer than that of the Riverview plates (usually 4 or 6 minutes during minima, sometimes 10 minutes) may also contribute slightly to the shallowing of the minima. Although the V plates and the Leiden plates were not used in forming the mean light curve, they were used to determine epochs of minima.

* Frances W. Wright, *H.A.*, **89**, 171 (10), 1937.
† *Riverview Publ.*, No. 1, 10, 1935.

The mean light curve is flat between minima, giving no indication of ellipticity of the stars or of a reflection effect. From 407 observations the magnitude at maximum was found to be $9^{m}\cdot892 \pm 0^{m}\cdot002$ (p.e.). The probable error of one estimate, obtained from the deviations of the individual estimates outside minima, is $\pm 0^{m}\cdot046$. Another value of the probable error was obtained from forty-eight pairs of exposures taken simultaneously with the R and G cameras, mostly during minima. The average difference between the estimates on two simultaneous exposures was $0^{m}\cdot079$. Hence the probable error of the difference between two estimates is $\pm 0^{m}\cdot067$. The resulting probable error for one estimate is $\pm 0^{m}\cdot047$, which agrees well with the other value. This is about the probable error usually found by the writer for estimates of eclipsing variables.

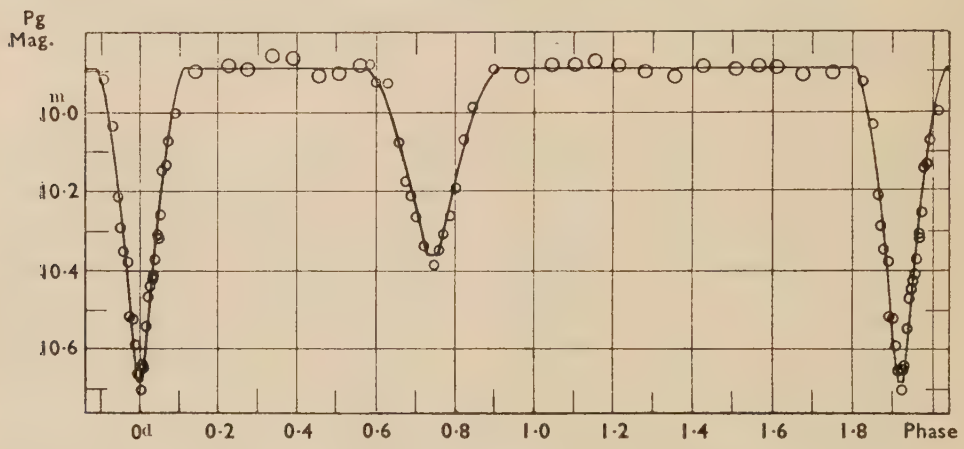

FIG. 1.—*Mean light curve of V 526 Sagittarii for epoch 4,200 (J.D. 2429924). Large circles denote means of 20, small circles of 10, estimates. The curve is that computed with the orbital elements in Table V.*

About 260 estimates on the X, R, P and G plates fall during primary minimum and 160 during secondary minimum. The observations during each minimum were divided into several intervals, each covering not more than about 1500 days. Mean light curves were formed for each interval, in order to investigate possible changes in the form of the light curve. There are slight indications of such changes (e. g. changes in the widths of the minima during the period covered by the observations), but the evidence is not conclusive. The question can be dealt with more easily at a later date, when more observations are available and when the changes should be more pronounced. In the meantime the combined observations during each minimum have been used to form a mean light curve.

The mean points are given in Table II. The phases in column 1 are in decimals of a day. In column 3 are given the number of estimates combined in each mean point, about ten during minima and twenty at maximum. The phases during minima were computed with the final formulae (4), given below, p. 340. For points outside minima the phases were calculated with the period $1^{d}\cdot919407$ and zero epoch J.D. 2429923·767, which is the time of primary minimum for epoch 4,200.

The mean points are plotted in Fig. 1. They represent the mean light curve for epoch 4,200 (J.D. 2429924). The light curve is that computed with the orbital elements derived in Section 5.

TABLE II

Mean Points

Outside minima				Primary minimum				Secondary minimum			
Phase	Mag.	No.	O–C*	Phase	Mag.	No.	O–C*	Phase	Mag.	No.	O–C*
d	m			d	m			d	m		
0·142	9·901	20	+ 9	−0·096	9·921	10	−11	−0·158	9·880	9	−15
0·227	9·886	20	− 6	−0·068	10·033	10	−45	−0·142	9·928	9	+ 8
0·273	9·895	19	+ 3	−0·055	10·214	10	+28	−0·108	9·928	10	−61
0·338	9·859	19	−33	−0·048	10·284	9	+36	−0·084	10·078	9	+ 8
0·389	9·866	20	−26	−0·041	10·347	10	+41	−0·065	10·176	9	+36
0·458	9·912	19	+20	−0·032	10·378	10	−31	−0·053	10·214	9	+24
0·508	9·905	19	+13	−0·024	10·516	10	+17	−0·040	10·262	9	+ 15
0·559	9·984	19	− 8	−0·020	10·521	10	−30	−0·022	10·342	10	+ 8
0·967	9·915	19	+25	−0·013	10·592	10	−26	+0·005	10·388	10	+28
1·045	9·883	20	− 9	−0·006	10·656	10	−20	+0·016	10·349	10	− 1
1·102	9·883	19	− 9	+0·002	10·701	10	+21	+0·029	10·305	11	+ 5
1·156	9·877	19	−15	+0·008	10·646	10	−20	+0·045	10·249	10	+26
1·217	9·885	19	− 7	+0·012	10·641	10	+10	+0·058	10·195	10	+29
1·280	9·899	20	+ 7	+0·018	10·545	10	−25	+0·078	10·077	10	− 8
1·352	9·916	19	+24	+0·022	10·473	10	−48	+0·105	9·987	11	−11
1·427	9·888	20	− 4	+0·027	10·445	10	−22	+0·154	9·897	11	− 3
1·508	9·898	19	+ 6	+0·030	10·424	10	− 2				
1·563	9·883	19	− 9	+0·034	10·412	10	+22				
1·613	9·887	19	− 5	+0·037	10·370	10	+22				
1·676	9·908	20	+20	+0·043	10·310	9	+17				
1·749	9·904	20	+12	+0·047	10·320	10	+67				
				+0·051	10·257	10	+38				
				+0·056	10·145	10	−27				
				+0·063	10·132	10	+18				
				+0·071	10·069	10	+ 8				
				+0·092	9·997	10	+51				

* Unit $0^m·001$

From the sum of the squares of the deviations of the mean points from the calculated curve the probable error of one mean point (mean of ten estimates) was found to be $\pm 0^m·021$ for primary, and $\pm 0^m·016$ for secondary, minimum. The resulting probable errors of one estimate are $\pm 0^m·067$ and $\pm 0^m·051$, respectively. The value for secondary minimum is not much greater than that found above ($0^m·046$) for estimates outside minima. The deviations during primary minimum are appreciably greater. It does not seem that they could be reduced much by an improvement in the orbital elements. The deviations during secondary minimum could be reduced by making the minimum slightly deeper, but this would entail either a change in the depth of primary minimum or an alteration in the orbital elements. In either case the resulting calculated curve would fit the observations during primary minimum less well. It does not seem worth while, at this stage, trying to improve the results by a least squares solution.

The beginning and end of both minima are less well determined than the other phases. Both minima are appreciably longer than was at first estimated, so that series of plates intended to cover the minima do not include the earlier and later phases. Further observations are needed. Meanwhile the material already available is quite good enough for a preliminary determination of the orbital elements.

5. *The Orbit.*—Secondary minimum is shallower and much wider than primary minimum and is of correspondingly less weight in determining the orbit. The shape of the light curve during primary minimum seems to demand as small a value of k, the ratio of the radii of the stars, as is compatible with the other data. On the assumption of uniformly illuminated disks the best fit is obtained with a total eclipse at primary minimum and an inclination of 90°. The magnitude outside eclipses was taken as $9^{m}\cdot89$, and at primary and secondary minimum as $10^{m}\cdot68$ and $10^{m}\cdot36$ respectively. k is then 0·85. A solution with total darkening to the limb suits the observed curve less well.

The eclipses could also be taken as partial, with magnitudes $10^{m}\cdot70$ and $10^{m}\cdot39$ at the minima, without doing too much violence to the observations. The observed depths of minima do not in any case allow much latitude for k and α_1, the degree of obscuration at primary minimum. The extreme values are $k=0\cdot85$, $\alpha_1=1\cdot0$; and $k=1\cdot0$, $\alpha_1=0\cdot9$.

Assuming spherical stars and central eclipses, with the primary eclipse total and $k=0\cdot85$, elements were derived, by the Russell-Shapley method *, from the light curve at each minimum separately, as if for a circular orbit. The following "circular" elements were obtained:—

	Primary minimum	Secondary minimum
Radius of larger star	$r_1'=0\cdot194$	$r_1''=0\cdot281$
θ at beginning of eclipse	$\theta'=21°\cdot0$	$\theta''=31°\cdot3$
Duration of eclipse	$0^{d}\cdot224$	$0^{d}\cdot333$
Duration of total (or annular) phase ...	$0^{d}\cdot011$	$0^{d}\cdot025$

Sterne's equations † were used to obtain the elliptical elements. Since the eccentricity is high and the minima rather wide, the more exact form of these equations was used, i. e. $r''/r'=(1+g\eta)/(1-g\eta)$, where $\eta=\frac{1}{3}(2+\zeta)$, $\zeta=\cos\frac{1}{2}(\theta'+\theta'')$, $g=e\sin\omega$, e is the eccentricity of the orbit, and ω the longitude of periastron (measured from the ascending node, in the direction of motion, or the orbit of the star eclipsed at primary minimum).

This gives $e\sin\omega=0\cdot1893$.

$e\cos\omega$ was obtained from the separation of the minima by the formula

$$2e\cos\omega=(t_2-t_1-\tfrac{1}{2}P)\pi/P,$$

where t_1 and t_2 are the times of primary and secondary minimum respectively, and P is the orbital period. This formula is sufficiently accurate for present purposes. At epoch 4,200, which was taken as the mean epoch of the observations on which the light curve is based,

$$t_2-t_1-\tfrac{1}{2}P=-0^{d}\cdot214.$$

Thus $e\cos\omega=-0\cdot175$.

Hence $e=0\cdot258$, $\omega=132°\cdot7$ (at epoch 4,200).

From Sterne's equations (30′) (again using the more exact form including η) r_1 is found to be 0·238.

The orbital elements are given in Table V. The light curve computed from these elements is shown in Fig. 1.

* H. N. Russell, *Ap. J.*, **36**, 53, 1912.
† T. E. Sterne, *Proc. Acad. Sci., Washington*, **26**, 36, 1940.

6. *Apsidal Motion.*—Mean light curves were computed for both minima and were used to obtain improved epochs of minimum. For primary minimum the observations were divided into intervals of about 1500 days with some overlapping, i. e. some observations were used in each of two successive intervals. In no case did the observations used for a mean epoch of minimum extend over more than about 1500 days. The observations during secondary minimum were divided into intervals of about 1000 days and a mean epoch was determined for each interval. No overlapping was used. A shorter interval was used for the mean epochs of secondary minimum, because the period was changing more rapidly than that of primary minimum, as can be seen from the later portions of the curves of the residuals.

The later residuals depart very definitely from the linear formulae given by Dugan and Wright. They are not, however, consistent with so short a period of apsidal revolution as the $24{,}000P$ suggested tentatively by these authors.

Zero epoch, where the curves for the two sets of residuals intersect, was taken 40 periods later than that given by Dugan and Wright. Secondary minimum has been for some years past much wider than primary, so that primary minimum at epoch 0 was a periastron eclipse. The longitude of periastron, ω, was then 90°, and the line of apsides was parallel to the line of sight. The displacement of secondary minimum shows that periastron is advancing.

Assuming the apsidal motion to be uniform, if T_1 and T_2 are the initial epochs of primary and secondary minimum (when ω was 90°, T_2 being half a period later than T_1), and t_1 and t_2 the times of the minima at epoch E, then t_1 and t_2 are given by the following formulae:—

$$\left.\begin{aligned} t_1 &= T_1 + EP + a\sin(Ex) - b\sin(2Ex), \\ t_2 &= T_2 + EP - a\sin(Ex) - b\sin(2Ex), \end{aligned}\right\} \qquad (1)$$

where P is the mean sidereal orbital period; $a = Pe/\pi$; $b = 3Pe^2/8\pi$; $x = 360° \times P/U$; and U is the period of apsidal revolution. These equations are correct * to the second power of e, assuming the inclination to be 90°.

Taking $e = 0{\cdot}26$ and $P = 1^d{\cdot}91941$, then $a = 0{\cdot}159$, $b = 0{\cdot}0155$.

It remains to determine x.

From (1)

$$t_2 - t_1 - \tfrac{1}{2}P = -2a\sin(Ex). \qquad (2)$$

Values of $(t_2 - t_1 - \frac{1}{2}P)$ were obtained from the times of the better determined minima and values of Ex were then obtained from (2) for various values of E. x was thus found to be 0·0103. The period of apsidal revolution is then 35,000 P, about 184 years.

The mean period that was found to fit best the curves of the residuals is $1^d{\cdot}9194115$. The residuals of the observed minima from the formulae

$$\left.\begin{aligned} \text{Primary minimum} &= \text{J.D. } 2421862{\cdot}145 + 1^d{\cdot}9194115E, \\ \text{Secondary minimum} &= \text{J.D. } 2421863{\cdot}105 + 1^d{\cdot}9194115E, \end{aligned}\right\} \qquad (3)$$

are given in Table III, $(O-C)_1$, and are plotted in Fig. 2.

* Exact equations, as well as approximations equivalent to those used here, are given by J. Stein, S.J., *Die Veränderlichen Sterne*, II, 230, 1924.

From (1) the formulae for the times of minima are now

$$\left.\begin{aligned}&\text{Primary}\\&\quad\text{minimum} = \text{J.D. } 2421862{\cdot}145 + 1^{d}{\cdot}9194115E + 0{\cdot}159\sin(0^{\circ}{\cdot}0103E)\\&\qquad\qquad - 0{\cdot}0155\sin(0^{\circ}{\cdot}0206E),\\&\text{Secondary}\\&\quad\text{minimum} = \text{J.D. } 2421863{\cdot}105 + 1^{d}{\cdot}9194115E - 0{\cdot}159\sin(0^{\circ}{\cdot}0103E)\\&\qquad\qquad - 0{\cdot}0155\sin(0^{\circ}{\cdot}0206E).\end{aligned}\right\} \quad (4)$$

The residuals from formulae (4) are also given in Table III, $(O-C)_2$.

TABLE III

Minima of V 526 Sagittarii

Primary minimum						Secondary minimum					
Epoch	J.D. hel.		Wt.	$(O-C)_1$	$(O-C)_2$	Epoch	J.D. hel.		Wt.	$(O-C)_1$	$(O-C)_2$
	2400000+			Unit $0^{d}{\cdot}001$			2400000+			Unit $0^{d}{\cdot}001$	
−3730	14702·661	H	−1	− 79	+5	−4330	13552·173	H	−1	+120	− 6
3180	15758·339	H	1	− 77	−5	3680	14799·786	H	−1	+115	+ 2
2720	16641·278	H	1	− 68	−6	3160	15797·868	H	1	+103	+ 3
2400	17255·500	H	1	− 57	−2	2720	16642·387	H	−1	+ 81	− 7
1750	18503·138	H	2	− 37	+3	2320	17410·143	H	1	+ 73	− 3
1210	19539·624	H	2	− 33	−5	1750	18504·176	H	1	+ 41	−17
760	20403·375	H	2	− 18	0	1210	19540·661	H	1	+ 44	+ 3
− 240	21401·475	H	1	− 11	−5	760	20404·367	H	−1	+ 15	−11
+ 330	22495·554	H	1	+ 3	−5	− 240	21402·455	H	2	+ 9	0
910	23608·829	H	1	+ 20	−1	+ 200	22246·984	H	1	− 3	+ 4
1320	24395·797	H	−1	+ 29	−1	910	23609·733	H	1	− 36	− 5
2169	26025·401	L	6	+ 53	+3	1370	24492·660	H	−1	− 39	+ 7
2990	27601·254	H	1	+ 69	+1	1900	25509·914	H	−1	− 73	−10
3135	27879·570	R	3	+ 70	−1	2155	25999·367	L	5	− 69	+ 1
3300	28196·282	H	1	+ 79	+4	2452	26569·416	L	3	− 85	− 6
3410	28407·414	R	5	+ 76	−1	2667	26982·091	R	3	− 84	+ 2
4161	29848·911	R	7	+ 94	+2	2860	27352·522	H	−1	−100	− 8
4750	30979·455	R	7	+105	+1	3216	28035·832	R	6	−100	+ 1
5141	31729·951	R	12	+111	−1	3300	28197·052	H	−1	−111	− 8
5527	32470·850	R	5	+118	−1	3697	28959·053	R	4	−116	− 3
						4250	30020·478	R	2	−126	0
						4728	30937·947	R	8	−136	− 1
						5205	31853·499	R	3	−143	0

In Table III are listed the epochs of minima determined by Dugan and Wright from Harvard plates, as well as those determined at Riverview and those obtained from the Leiden plates. They are denoted in column 3 by the letters H, R and L respectively. The weights assigned by Dugan and Wright to their minima * are given in the table. Weights proportional to the number of observations on which each Riverview minimum is based were assigned to these minima. In order to make the weights of the Riverview minima comparable to those of Dugan and Wright, an estimate was made of the number of observations corresponding to their unit of weight. They used in all 143 observations during minima to determine 28 epochs of minimum. If we take their weight − 1 as equivalent to $\frac{1}{2}$, and assume that

* R. S. Dugan and F. W. Wright, *Princeton Cont.*, No. 19, 8, 1939.

their weights are roughly proportional to the number of observations used, it follows that their weight 1 corresponds to six observations. Accordingly, the Riverview minima were weighted on the basis of unit weight for six observations. This holds directly for the secondary minima. For the primary minima, as explained above, many observations were used twice over. In order to allow for this overlapping, the weights of the epochs of primary minimum were calculated on the basis of unit weight for ten observations. The Leiden minima are shallower and less well determined and a somewhat smaller weight was assigned to them (unit weight for eight estimates).

In Fig. 2 are drawn the curves $\pm 0{\cdot}159 \sin(0°{\cdot}0103E) - 0{\cdot}0155 \sin(0°{\cdot}0206E)$, which represent the differences between formulae (4) and (3) for each minimum.

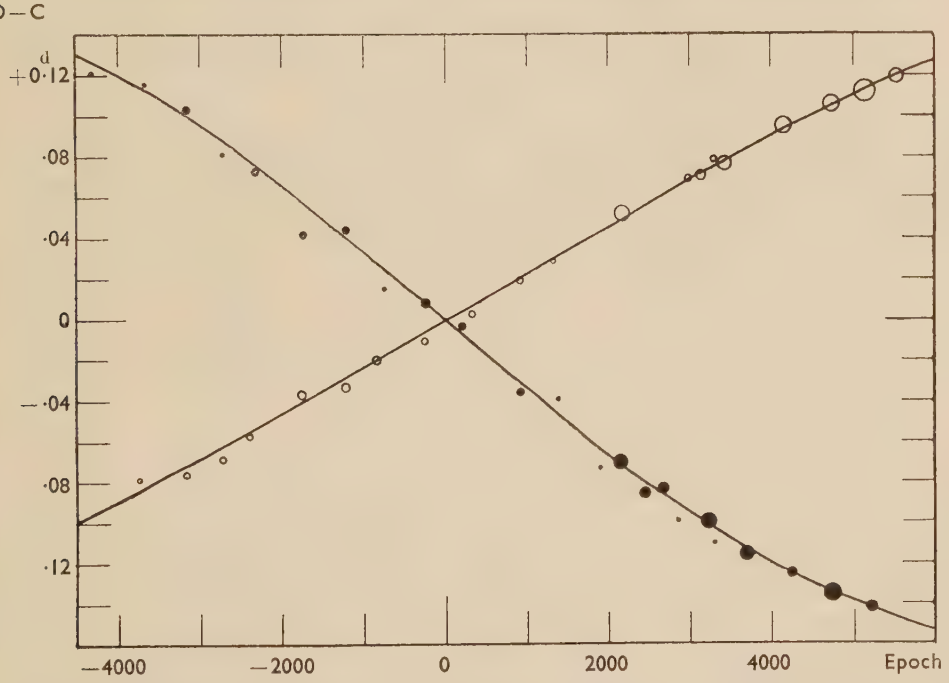

FIG. 2.—*Minima of V 526 Sagittarii. Abscissae are epochs, ordinates residuals from formulae* (3). *Open circles denote primary, closed circles secondary, minima. The curves represent the computed residuals,* $\pm 0{\cdot}159 \sin(0°{\cdot}0103E) - 0{\cdot}0155 \sin(0°{\cdot}0206E)$.

Primary minima are represented by open, secondary minima by closed, circles. The diameter of the circles is roughly proportional to the square root of the weight.

It is considered that e must be fairly close to the value derived here, 0·26, and consequently that the period of apsidal revolution cannot differ much from $35{,}000P$. A somewhat smaller value of e would suit the light curve fairly well, but not a larger one. On the other hand, the observed times of minima could be satisfied by a larger value of e, but not by one appreciably smaller. This may be seen from Table IV, in which are given, for different values of e, (1) the period of apsidal revolution; (2) the date at which the computed residual curve for secondary minimum reaches its maximum; and (3) the maximum value of $(t_2 - t_1 - \frac{1}{2}P)$, with the year when it occurs.

The trend of the residual curves between epochs 2,000 and 5,500 is pretty well defined by the observations. The curve for secondary minimum can hardly

reach its turning-point as early as epoch 6,900. Thus so low a value of e as 0·24 seems excluded.* It is probable that the value of 0·26 for e is correct to one unit in the second decimal place. In any case, observations over the next five or six years will probably settle the question conclusively.

TABLE IV

Eccentricity, e		0·24	0·26	0·28	0·29
Period of apsidal revolution	Epochs	31,000	35,000	38,500	40,000
Maximum of residual curve for	Epoch	6,920	7,720	8,460	8,700
secondary minimum	Year	1955	1959	1963	1964
Maximum of $(t_2-t_1-\frac{1}{2}P)$	Amount	0^d·294	0^d·318	0^d·342	0^d·354
	Year	1959	1964	1969	1971

TABLE V

Summary of Results

Mean period, P	1^d·9194115
Eccentricity of orbit, e	0·26
Longitude of periastron, ω	90° in 1918
Period of revolution of apsides	35,000P (184 years)
Direction of apsidal motion	Direct

Uniform Solution

Primary eclipse total	
Inclination of orbit plane, i	90°
Radius of large star, r_1	0·238
Radius of small star, r_2	0·202
Light of large star, L_1	0·483
Light of small star, L_2	0·517
Ratio of surface brightnesses, J_2/J_1	1·5
Density of large star, ρ_1	0·13 $\odot$
Density of small star, ρ_2	0·22 $\odot$

Formulae (4) should be used to predict times of minima.

The results are summarized in Table V. The densities of the stars were computed on the assumption of equal masses. The uniform solution only is given. This is considered to be merely a preliminary orbit determination and it did not appear worth while to attempt further refinements, such as determining the degree of darkening to the limb. Finer details of the light curve, and variations in its form, are obscured by the fact that the observations of several years were combined to determine the curve. Photoelectric observations would provide very valuable information; each minimum can be covered in a single night. Spectrum observations, light curves in different wave-lengths and radial velocity measures are badly needed for this interesting binary.

Acknowledgments.—The writer is indebted to Professor Oort for permission to work on the Leiden plates; and to Father J. de Kort, S.J., of the Vatican Observatory, for helpful criticisms and suggestions.

Riverview College Observatory,
Riverview, N.S.W.:
1948 *April* 23.

Specola Vaticana,
Castel Gandolfo:
1948 *July* 12.

* As pointed out above, the formulae used here are approximate. Father J. de Kort, S.J., in an unpublished paper, derives the following exact equation for finding the epoch of maximum of the residual curve for secondary minimum (for an inclination of 90°): $1-e\cos Ex=(1-e^2)^{\frac{1}{2}}$. With this equation the epochs in line 3 of Table IV become 6850, 7660, 8320, 8600. These are slightly less than the values given in the table. Thus these more exact results strengthen the conclusion that e is greater than 0·24.

THE PHOTOMETRIC ELEMENTS OF RS SAGITTARII

R. L. Baglow

(Communicated by the Director of the Cambridge Observatories)

(Received 1948 July 16)

Summary

Photometric elements have been derived from Redman's observations of this star. Although the measures are accurate, the calculated dimensions of the system are subject to a wide range of uncertainty, which could however be considerably reduced by suitable spectroscopic observations.

Material.—The observations were made by R. O. Redman between 1939 August and 1940 April, their mean epoch being J.D. 2429542. The measurements were made photographically by the Fabry method and refer to an effective wave-length 4400 A.* Figs. 1 and 2 show the observations, together with a

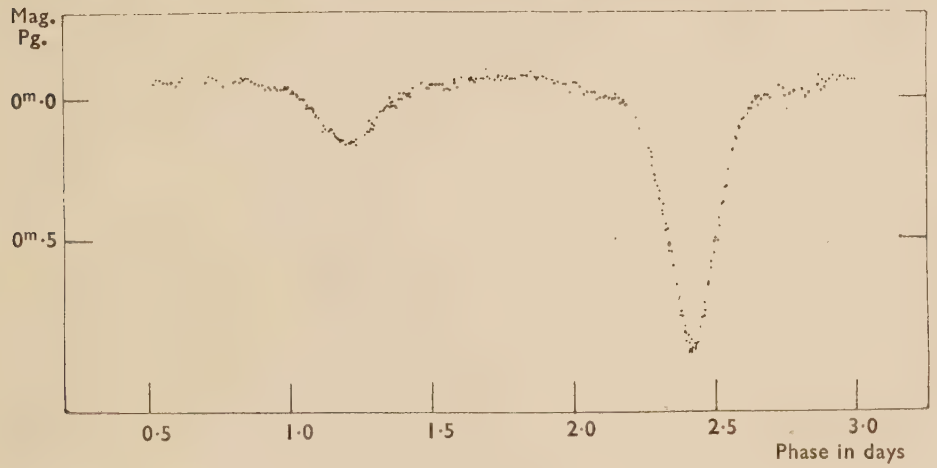

FIG. 1.—*Complete light curve.*

theoretical curve for both minima, computed as described below. Each plotted point represents the mean of four measures, except at the bottom of primary minimum, where each is the mean of about eight or ten measures. The observed light curve shows no asymmetry that cannot be explained by the uncertainties of observation. The orbit is circular.

Rectification.—For the purpose of rectification, a smooth curve was drawn through a plot of the observations, and normals were taken at intervals of 0·02 days. The normals at equal phases before and after principal minimum were meaned. Harmonic analysis of 31 normals yielded

$$l = \underset{\pm 0\cdot0005}{0\cdot9553} - \underset{\pm 0\cdot0004}{0\cdot0271}\cos\theta - \underset{\pm 0\cdot0005}{0\cdot0457}\cos 2\theta.$$

---

* R. O. Redman, *M.N.*, **105**, 212, 1945.

The light curve was rectified by methods described by Russell.* The spectral type of the brighter star is given as B5 and that of the fainter has not yet been observed. It was assumed to be of type A0 for the purpose of calculating the correction for reflection. Following Russell, Z ($=\epsilon \sin^2 i$) was taken as 0·046.

Analysis.—Full tables of functions for determining the elements of eclipsing systems were not available.† Tables were therefore calculated for various degrees of darkening from those given by Russell and Shapley.‡ The same degree of darkening was assumed for both stars. The eclipse of RS Sagittarii is a deep partial, and is an unfavourable case for the determination of limb darkening. Various degrees of darkening were tried, but the effect on the light curve was too small to allow the coefficient to be determined. The elements were finally

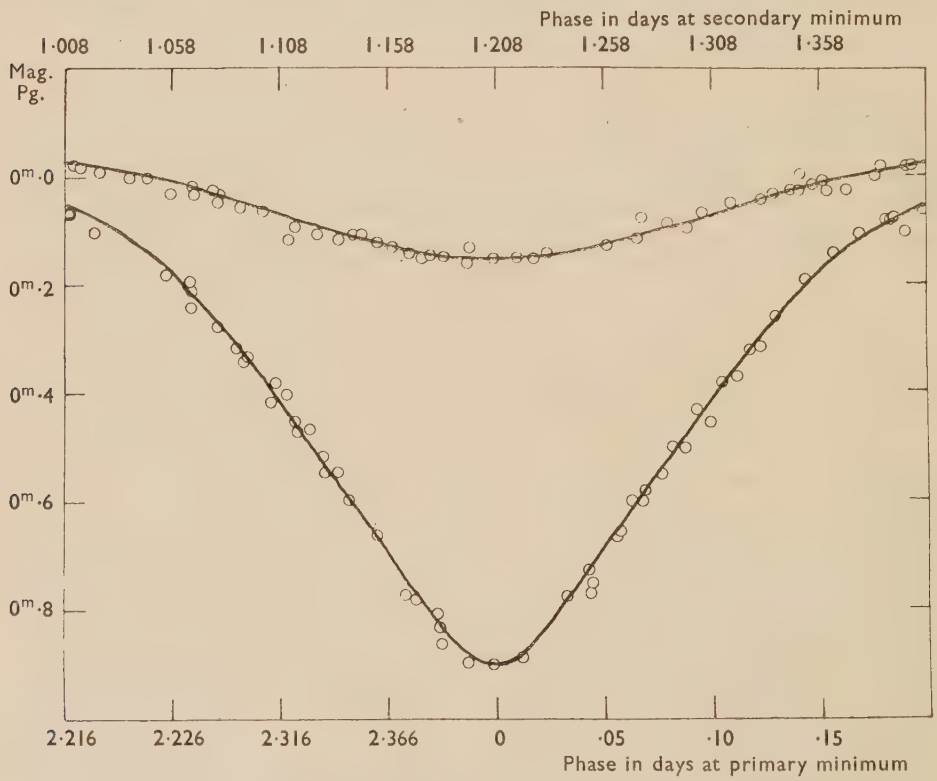

FIG. 2.—*Primary and secondary minima. Observations compared with computed curve.*

computed for an assumed coefficient of limb darkening $u = 0{\cdot}4$, by the method described by Kopal.§ The light curve can be fitted well with a wide range of the ratio of the radii. If r_1 is the radius of the brighter star, r_2 of the fainter,

$$0{\cdot}70 < r_1/r_2 < 1{\cdot}33.$$

The adopted value of k should give a linear plot of $\sin\theta(n)$ against $k(P^2 - P_0^2) + 2(P - P_0)$; at the limiting values the curvature of this graph is beginning to become too pronounced to be permissible.

* H. N. Russell, *Ap. J.*, **104**, 153, 1946.
† Zessewitsch, *Bull. Astr. Inst. U.S.S.R. Acad. Sci.*, No. 45, 1939, and No. 50, 1940.
‡ H. N. Russell and H. Shapley, *Ap. J.*, **36**, 239, 1912.
§ Z. Kopal, *Harvard Observatory Monograph*, No. 6, 1946.

Results.—

TABLE I

RS Sagittarii	R.A. $18^h\ 11^m$	$\delta\ -34°\ 08'$ (1900)
	Maximum light	$6^m{\cdot}0$
	Primary minimum	$6^m{\cdot}97$
	Secondary minimum	$6^m{\cdot}22$
	Period	2·41568 days

From the graphical determination

$$r_1/r_2 \sin^3 i = 0{\cdot}0825 \pm 0{\cdot}001 \text{ (estimated)},$$

in units of the orbital radius. It will be seen that the orbital elements are rather indeterminate. The following Table II gives alternative elements and it is thought unlikely that the true elements lie outside the range considered.

TABLE II

r_1/r_2	0·75	1·0	1·25
P_0	−0·550	−0·377	−0·743 (depth of eclipse)*
$\cot i$	0·179	0·179	0·130
r_1	0·325	0·283	0·318
r_2	0·391	0·283	0·255
L_2/L_1	0·391	0·243	0·169 (ratio of luminosities)
J_2/J_1	0·15	0·14	0·11 (ratio of surface brightnesses)

* If δ is the separation of centres at mid-eclipse, $\delta = r(1 + kP_0)$, (see *Ap. J.*, **36,** 239, 1912).

The solution giving the larger value of r_1/r_2 is favoured, for two reasons. The first is that to give a calculated value of the reflection coefficient as small as 0·027 (the coefficient of $\cos\theta$ above), r_2 must be smaller than r_1.* The second reason for favouring the solution with the smaller r_2 is that if the system is to conform to the mass-luminosity relation it appears that the secondary must be considerably less massive than the primary.

Taking the radial velocities given in the *Publications of the Lick Observatory* † we get for the velocity of the B5 star in a circular orbit 70 ± 1 km./sec. This leads to a value of the radius of the orbit of the B5 star about the centre of gravity of the system of $2{\cdot}33 \times 10^6$ km. Unfortunately, until the spectrum of the fainter star is observed the absolute dimensions of the system cannot be found, nor can the observed ellipticity of the components be compared with that which is to be expected theoretically. The following Table III shows the dimensions of the system for certain assumed values of the mass ratio. The magnitudes given were calculated for an assumed temperature of the B5 star of 15,000 deg., taking the radius as 0·283, and for comparison with the mass-luminosity curve the bolometric correction given by Eddington in *The Internal Constitution of the Stars* was applied.

TABLE III

Assumed mass ratio	1·0	0·5	0·2
Radius of relative orbit	4·7	7·0	14×10^6 km.
Sum of masses	0·7	2·4	$19\odot$
r_1	1·3	2·0	4×10^6 km.
Absolute luminosity of B5 star (bolometric)	$+0^m{\cdot}6$	$-0^m{\cdot}4$	$-1^m{\cdot}9$

The first two columns give a luminosity of the B5 star which is improbably low.

* H. N. Russell, *Ap. J.*, **102,** 6, 1945.
† *Publ. Lick Obs.*, **16,** 265, 1928.

RS Sagittarii appears to be a normal eclipsing binary, with a circular orbit, and a secondary star probably considerably less massive than the primary. The stars are of comparable size, with the secondary probably rather the smaller of the two. The large uncertainties in the photometric results could be much reduced by a spectroscopic study of the system, and this is much to be desired.

In conclusion it is a pleasure to express my thanks to Professor R. O. Redman for suggesting this problem and for aid and counsel in its examination.

The Observatories,
Cambridge :
1948 *July* 14.

ON AN APPARENT CHANGE OF WAVE-LENGTH IN SOLAR IRON LINES

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(Received 1948 August 3)

Summary

The telluric oxygen lines in the solar spectrum have been used to test the stability of four solar iron lines, 6280, 6297, 6301 and 6302, interspersed within the α group. In the interval of about twenty years the absolute wave-length of each of these lines photographed at the centre of the Sun's disk appears to have decreased by measurable amounts. 6280 representing a low level in the reversing layer, is the least affected, decreasing by only 0·002 A., whilst the line 6301 representing a relatively high level, has decreased by 0·008 A. Table I shows the various yearly values of wave-lengths obtained, and the differences from the " Recommended " values of the International Astronomical Union.

At the Sun's limb, the wave-lengths of the four lines appear also to have diminished. In Table II the increase of wave-length at the limb compared with the centre spectra is shown, and this quantity for the lines 6301 and 6302 appears not to have changed materially from the earlier measures of Halm and Adams.

An investigation of the constancy of wave-lengths of atmospheric and solar lines was made by St John and Babcock * with the result that they found no reason to question the stability of the telluric lines of oxygen in the B group, and in the α group in spectra taken at intervals from sunrise to sunset, and in different years, indicating the absence of high-velocity currents in the Earth's atmosphere.

Of the constancy of the solar lines at the centre of the Sun's disk they also found that, taking the means of groups of from 13 to 21 lines in different regions of the blue, no appreciable changes were found in the years 1917 and 1921.

On the other hand, when investigating the constancy of wave-length of individual lines in "sunlight" spectra, that is light mainly derived from the central region of the Sun's disk, ruling out any local disturbances which might affect a small region at the centre, I found some marked variations amounting to several thousandths of an angstrom in the region 4337–4531, using for comparison lines in the iron arc not subject to " pole effect."† A. Hunter ‡ has called attention to an apparent change in the solar "limb effect", that is the increased wave-length of lines at the Sun's limb as compared with those at the centre of the disk, in the region 6280–6322. The redward shift at the limb appeared to have changed from +0·012 in the older measures of Halm in 1907 and Adams in 1910, to +0·008 in 1932. This may indicate a change of wave-length at the limb, or at the centre, or both.

These results and the marked instability of the sodium line D_1 at the centre of the Sun, shown in recent measures of this line § suggested the desirability of testing the stability of the stronger iron lines representing relatively high levels in the reversing layer of the Sun. The α group of telluric oxygen lines form ideal

* C. E. St John and H. D. Babcock, *Ap. J.*, **55**, 36, 1922.

† *Annual Report Kodaikanal and Madras Observatories*, 1919.

‡ A. Hunter, *M.N.*, **94**, 594, 1934.

§ " New Measures of the Sodium Line D_1 in the solar spectrum," *M.N.*, **105**, 200, 1945.

standards of reference for testing the stability of the solar lines interspersed among them. I therefore undertook, between 1940 and 1948, to make an extended series of measures in this region. The spectra were obtained with the liquid prism spectrograph, using the equivalent of ten or fifteen prisms by multiple transmission through two prisms and one reflecting half prism. The linear dispersion varied from about 1 mm./A. to 1·5 mm./A. The spectra were measured by two methods, those of lower dispersion but having the best contrast mainly by the positive-on-negative method; the rest by an accurately engraved glass plate fixed above and almost in contact with the negative to be measured, instead of the usual scheme where it is placed in the eyepiece of the microscope. The negative is attached to the long nut of the micrometer screw, and moves with it—a great improvement on the old scheme of dragging the microscope along oiled ways over a fixed negative.*

The graphical method of the reduction of the measures has been adopted; it is sufficient to state here that the wave-length of each one of the four iron lines measured in this region was determined from an oxygen line on each side of it. These O lines are based on interferometer comparisons with neon lines, and assuming that these ultimately depend on the cadmium standard, absolute values of the solar iron lines are obtained, after correcting for the orbital and diurnal motions of the Earth. These values may therefore be compared with the wave-lengths of iron lines in terrestrial sources for finding the gravitational effect of Einstein.

Unfortunately these particular lines do not appear to have been measured in the vacuum arc with sufficient accuracy for this comparison; but in any case the telluric lines, whatever wave-lengths are assumed for them, can be used to find any changes in the solar lines, assuming only their constancy, which appears to be justified.

In the year 1940, I had already measured four spectra in this region, and obtained marked deviations from the "Recommended" wave-lengths published in *Trans. I.A.U.*, **3**, 99, 1928. In the later years, I have confined attention to the following four solar iron lines, 6280, 6297, 6301 and 6302, using the O lines more or less symmetrically placed on either side of each, and using also a series of five O lines from 6279 to 6306 for determining the varying scale of the plates. I have omitted the strong *Fe* line 6290, as it is not well-enough defined in most spectra to admit of very accurate measurement. It is perhaps necessary to state that the Sun's image, of about 60 mm. in diameter, was accurately centred on the middle point of a shortened slit, and would not deviate from that position appreciably even during the comparatively long exposures of one or two minutes required for the 15-prism spectra.

The line 6280, intensity 2, excitation potential 0·855, probably represents a low level in the reversing layer, and the other three a higher level, especially the line 6301, intensity 7, e.p. 3·6, and this is confirmed by the fact that the former always gives a rotation value about 3 to 5 per cent smaller than the others in measures of limb spectra, in accordance with the remarkable rule that the angular speed of rotation increases with height above the photosphere.

In the table which follows I give in four columns the fractional wave-lengths of the lines measured, beginning with the "Recommended" wave-lengths of the

* For a full description of the micrometer and the method of reduction of the measures, see *M.N.*, **99**, 120 and 122, 1938.

I.A.U., and followed by my recent measures of two old Kodaikanal grating spectra of a central section across the whole disk of the Sun. These are in very close agreement with the "Recommended" values. Next follows the mean of three Edinburgh grating spectra, photographed by Storey in the year 1934, and kindly sent to me by Dr Ellison. Following this are the yearly mean values of the Ewhurst prism spectra from 1940, with the means of each of the four years up to 1948 May 18.

TABLE I

Decrease of Wave-length of Solar Iron Lines

Year	Source of the Measures	6280	6297	6301	6302
1928	International Astronomical Union	0·622	7·799	1·508	2·499
1914	Kodaikanal from a paper print	0·620	7·785	1·508	2·496
1915	Kodaikanal from a negative	0·622	7·799	1·508	2·498
1934	Edinburgh mean of 3 spectra	0·622	7·795	1·504	2·497
1940	Ewhurst mean of 4 spectra	0·623	7·796	1·503	2·497
1946	Ewhurst mean of 43 spectra	**0·620**	**7·793**	**1·500**	**2·495**
1947	Ewhurst mean of 14 spectra	0·622	7·794	1·501	2·496
1948	Ewhurst mean of 11 spectra	0·621	7·795	1·502	2·498
Decrease of wave-length of the 1946 spectra		−0·002	−0·006	−0·008	−0·004

It appears that the minimum wave-lengths for all the four lines was reached in 1946, and this is followed by a tendency to increase. These minimum values are in heavy type in the table, and the differences from the International values are given at the foot of the columns. The low-level line 6280 is the least affected, having diminished by two units only in the third decimal, while the high-level line 6301 decreases by eight units. In the year 1947 August, I was able, with the cooperation of Dr Thackeray, to secure two grating spectra at the Cambridge Observatory and measures of these confirm very closely the results of the prism spectra, giving an even lower value for the line 6301.

It is unfortunate that I have only two spectra from earlier years to compare with the recent values, but these agree closely with the International values. It must be stated, however, that the interferometer measures of these lines, given in the last column of Table IV of Vol. 3 of the *Trans. I.A.U.* differ slightly from the "Recommended" values, and if these are substituted the decrease would be less, except for the first line, as follows :—

6280	6297	6301	6302
−0·003	−0·005	−0·005	−0·002

As regards the accuracy of the measures, the mean yearly values are not likely to be in error by as much as one unit in the third decimal, except perhaps the 1940 measures of only four spectra. The low-level line 6280 is remarkably constant in wave-length, and has therefore a smaller uncertainty than the others, which occasionally show rather large changes from plate to plate, indicating real changes. The line 6297 is the best for the accuracy of measurement.

General shift to red of the lines at the Sun's limb

I now give some results of measures of limb spectra showing the general increase of wave-length at the limb compared with the centre of the disk. This increase can be measured directly when the spectra have been photographed at the same time both at the centre and at the limb. This has been done in the case of the Edinburgh spectra of 1934 already alluded to. These photographs include, in addition to the three spectra of the centre of the disk, three of the east limb and three of the west limb. The results are given in Table II, following the earlier measures of Halm in 1907 and Adams in 1909 for two of the lines. In the case of the Ewhurst spectra obtained in subsequent years, the east and west limbs were photographed simultaneously by the method described in *M.N.*, **93**, 165, 1933, whilst the centre of disk spectra were obtained separately, and at different times. The true wave-lengths of both the limb spectra and the centre spectra were determined by applying the appropriate corrections for the orbital and diurnal motions of the Earth. The centre wave-lengths were then subtracted from the limb wave-lengths; and as the differences are liable to rather wide changes on different dates, due probably to bad definition of the Sun's limb, I have in Table II given the means of several spectra in the different years.

For all the limb spectra from 1940 to 1943 inclusive I used the very consistent values obtained in the centre spectra in 1940 and in the 1946 series I used the slightly smaller wave-lengths obtained from 43 centre spectra in that year.

TABLE II

Mean Shifts, Limb—Centre

Year	Source of the Measures	6280	6297	6301	6302
1907	Halm			+0·012	+0·012
1909	Adams 0·99 from centre			+0·012	+0·012
	No correction to limb applied				
1934	6 Edinburgh spectra 0·97 from centre	+0·008	+0·010	+0·011	+0·009
	Correction to limb +0·002				
1940	20 Ewhurst spectra 0·97 from centre	+0·007	+0·008	+0·011	+0·009
	Correction to limb +0·002				
1941	18 Ewhurst spectra 0·94 from centre	+0·007	+0·009	+0·011	+0·010
	Correction to limb +0·003				
1943	6 Ewhurst spectra 0·94 from centre	+0·008	+0·006	+0·010	+0·010
	Correction to limb +0·003				
1946	20 Ewhurst spectra 0·98 from centre	+0·007	+0·010	+0·011	+0·008
	Correction to limb +0·001				

It is very necessary in all these measures to correct for the distance within the limb at which the spectra were secured. In Table II these corrections have been applied to the Ewhurst and Edinburgh measures: they are estimated from my measures of the change of wave-length between centre and limb, given in Kodaikanal Observatory Bulletin No. 49. The mean results for the four years of the

Ewhurst spectra, when so corrected, are in better agreement than was anticipated, and those of the year 1946 agree very closely with the mean of the three Edinburgh spectra of the year 1934 measured directly from the centre spectra impressed on the same plate.

As to the earlier results of Halm and Adams, there appears to be only a slight decrease in the later years, but if I had used the "Recommended" values of the I.A.U. for the centre of the disk, the limb-centre shifts would have been very much smaller.

These results of the limb spectra appear to show that the wave-lengths at the limb must also have diminished, if anything, even slightly more than at the centre.

Highbroom,
Ewhurst, Surrey :
1948 *July* 30.